Description of 3D Morphology using the Concept of the Divider Set

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Abstract

We describe a novel concept for classification of complex 3-dimensional geometry based on a concept we refer to as the Divider Set. It is a novel alternative concept to maximal disks, Voronoi sets and cut loci, which is based on a formal definition relating to topology and differential geometry. In this paper we introduce the concept of the Divider Set within the context of definition morphology of objects. We then discuss the computation the Divider Set for complex 3-dimentional geometry. In particular, in this paper, we have shown how the Divider Set can be computed for surfaces described in parametric form. In order to computer the Divider Set, two forms of solutions have been described, one analytic which takes advantage of the special parametric form of the surface and the other a numerical solution which can be utilised for general parametric surfaces. In order to show the applicability of the techniques we illustrate our concepts through a number of examples.

Key words: divider, skeleton, medial axis, morphology, computational geometry.

1 Introduction

Mathematical morphology is a technique for representation and description of complex objects in a simpler form by means of providing a quantitative description of geometrical structure of the object in question. Morphology can provide information regarding the boundaries of an object, its medial axis, skeletons or Voronoi sets, and its convex hull. Generally speaking most morphological methods are based on simple expansion and shrinking operations. These techniques have many practical applications. These include, image processing [9], shape similarity studies [5] and pattern recognition [8].

The most commonly used morphological description for geometric shapes is based upon the medial axis or the skeleton. Starting from the pioneering paper of H. Blum [5], the medial axis, as a descriptor and classifier of shapes and figures, has been established as the best defined and studied mathematical concept with reference to thinning and skeletonization of contours and shapes. From the various mathematical tools (e.g. maximal disks, cut loci, Voronoi sets [19,7]), the maximal disk method seems to be the most well studied and applied, both in mathematical definition and properties and in applications.

The definition given by Blum is best presented in the following form,

Definition 1: Let S be a closed contour in $\Re^2$. A closed disk B is said to be *maximal* in S if it is contained in S and if $B \subset B'$, where B' is another closed disk, also contained in S, then $B = B'$ [19,7]. The notion of maximal disks is based on the Euclidean metric,

$$d_E(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{1}$$

Based on the definition, one can see that the medial axis construction results in an object which neatly describes the morphology and geometric characteristics of the underlying shape. Furthermore, through the medial axis it is generally easier to identify the symmetries of the object. Other important properties of the media axis of a shape include its use in the intermediate representation of the object and its canonical general form that can be used to represent the object by a lower dimensional description. It is notable that many others have affirmed the flexibility of the medial axis and its ability to naturally capture important shape characteristics of an object [16,13,6].

There a number of great many references on the medial axis based on the

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above definition. For comprehensive surveys the reader is referred to [11,14]. A number of methods for constructing medial axis or skeletons for polyhedral models and for free-form shapes have also been proposed. These include topological thinning [20], Euclidian distance transform[1] and the use of deformable snakes [12]. For examples of practical algorithmic implementation of medial axis transforms the reader is referred to [15,17].

In a completely different field and with entirely different motivations, some similar concepts relating to the description of curvature of locally convex types were developed by Y. Bakopoulos and P. C. Stavrinos in the 80's [2]. The authors were attempting to develop better tools for the classification and extraction of features of various geometric constructions, such as classes of two dimensional manifolds immersed in a three dimensional Euclidean space. Such methods have applications in various branches of mathematics and physics, for example in knot theory, convexity, flows, the study of differential equations and the propagation of their solutions and corresponding singularities, as described by the Huygens principles [18].

This paper describes a new concept for classification of complex geometry based on the concept we call as the Divider Set. It is a novel alternative concept to maximal disks, Voronoi sets and cut loci, which is based on a formal definition based on topology and differential geometry. In particular, here our emphasis is placed on the computations of the Divider Set for complex 3-dimentional geometry particularly for surfaces described in parametric form. The paper is organised as follows. In Section 2 we introduce the concept of the Divider Set along with formal definitions. In Section 3 we discuss how we can formulate solution techniques for computing Divider for parametric surfaces which can be described in a simpler form with an associated analytic expression. In Section 4 we discuss a numerical solution technique for computation of the Divider for complex surfaces, in particular parametric surfaces which are polynomial in nature. Finally in Section 5 we conclude the paper and discuss possible extensions of the current work.

2 The Concept of the Divider Set

The medial axis of a closed contour in $\Re^2$, in mathematical morphology, is probably the closest equivalent to the definition of the Divider Set in $\Re^2$ or $\Re^3$. The formal definition of the Divider Set in $\Re^2$ or $\Re^3$ is as follows.

Definition 2: Let $q \in S \subset \Re^2$ or $\Re^3$. Let $p \in \Re^2$ or $\Re^3$ define a direction $\mid pq \mid$ in $\Re^2$ or $\Re^3$ such that a closed ball $B(p, \mid pq \mid)$ having p as the centre and $\mid pq \mid$ as the radius, in $\Re^2$ or $\Re^3$, with q as the only point with S. i.e. $B(p, \mid pq \mid) \cup S = q$. Let $B_c(p_c, \mid p_c q \mid)$ be the supremum of all balls $B(p, \mid pq \mid)$

with the above properties. Then, $B_c(p_c, \mid p_c q \mid)$ is defined as the maximum contact ball of S at q. $\frac{1}{|p_c q|} = k_c$ is defined as the contact curvature of S at q and the locus of all p_c's for all q's of S is called the Divider of S.

In the form mentioned above, the definition of the Divider in some sense is equivalent to that of the maximal disks, Voronoi sets and cut loci. However, one should note that there is a whole mathematical background to the Divider concept, developing the above definition into a set of equations, as shown below. The mathematical definition gives the Divider concept specific advantages over the other definitions available for describing morphology. Thus, all definitions pertaining to medial axis, Voronoi sets, cut loci or any such concepts are replaced by a common definition expressed in a strict equation form. The defining equations can be adapted to apply for curves [2] or disconnected sets of curves in $\Re^2$ or $\Re^3$, for isolated points, for surfaces in $\Re^3$ or for a mixture of the above. Furthermore, by a suitable change of the fundamental metric in the Euclidean plane and by considering a discretization of the plane as in binary images, an entirely new form of the Divider can be developed and utilized in various applications.

In this paper the defining equations for the Divider is adapted to the three dimensional case whereby we consider surfaces $S \in \Re^3$ with some specific properties. The surfaces we consider are essentially parametric and should be, except for isolated points or curves, twice continuous and differentiable.

Let a surface $S \in \Re^3$ be defined by the parametric form using two parameters u and v such that $S(u, v) = (x(u, v), y(u, v), z(u, v))$. Given any values u_1, v_1 and u_2, v_2 of the defining parameters of S we can define two points on S such that,

$$X_1(u_1, v_1) = (x_1(u_1, v_1), y_1(u_1, v_1), z_1(u_1, v_1)), \tag{2}$$

$$X_2(u_2, v_2) = (x_2(u_2, v_2), y_2(u_2, v_2), z_2(u_2, v_2)). \tag{3}$$

Thus, taking three points $(x, y, z), (x_1, y_1, z_1), (x_2, y_2, z_2) \in S$ the calculation of the Divider involves solving the following equations.

$$(x_1 - x)\frac{\partial x_1}{\partial u_1} + (y_1 - y)\frac{\partial y_1}{\partial u_1} + (z_1 - z)\frac{\partial z_1}{\partial u_1} = 0, \tag{4}$$

$$(x_2 - x)\frac{\partial x_2}{\partial u_2} + (y_2 - y)\frac{\partial y_2}{\partial u_2} + (z_2 - z)\frac{\partial z_2}{\partial u_2} = 0, \tag{5}$$

$$(x_1 - x)\frac{\partial x_1}{\partial v_1} + (y_1 - y)\frac{\partial y_1}{\partial v_1} + (z_1 - z)\frac{\partial z_1}{\partial v_1} = 0, \tag{6}$$

$$(x_2 - x)\frac{\partial x_2}{\partial v_2} + (y_2 - y)\frac{\partial y_2}{\partial v_2} + (z_2 - z)\frac{\partial z_2}{\partial v_2} = 0, \tag{7}$$

$$(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2 = (x_2 - x)^2 + (y_2 - y)^2 + (z_2 - z)^2. \quad (8)$$

Equations (4) - (8) can also be re-written in the following format,

$$x\frac{\partial x_1}{\partial u_1} + y\frac{\partial y_1}{\partial u_1} + z\frac{\partial z_1}{\partial u_1} = x_1\frac{\partial x_1}{\partial u_1} + y_1\frac{\partial y_1}{\partial u_1} + z_1\frac{\partial z_1}{\partial u_1}, \tag{9}$$

$$x\frac{\partial x_2}{\partial u_2} + y\frac{\partial y_2}{\partial u_2} + z\frac{\partial z_2}{\partial u_2} = x_2\frac{\partial x_2}{\partial u_2} + y_2\frac{\partial y_2}{\partial u_2} + z_2\frac{\partial z_2}{\partial u_2}, \tag{10}$$

$$x\frac{\partial x_1}{\partial v_1} + y\frac{\partial y_1}{\partial v_1} + z\frac{\partial z_1}{\partial v_1} = x_1\frac{\partial x_1}{\partial v_1} + y_1\frac{\partial v_1}{\partial v_1} + z_1\frac{\partial z_1}{\partial v_1}, \tag{11}$$

$$x\frac{\partial x_2}{\partial v_2} + y\frac{\partial y_2}{\partial v_2} + z\frac{\partial z_2}{\partial v_2} = x_2\frac{\partial x_2}{\partial v_2} + y_2\frac{\partial y_2}{\partial v_2} + z_2\frac{\partial z_2}{\partial v_2}, \tag{12}$$

$$\begin{aligned} &{x_1}^2 - {x_2}^2 + {y_1}^2 - {y_2}^2 + {z_1}^2 - {z_2}^2 = 2((x_1 - x_2)x + (y_1 - y_2)y \\ &+(z_1 - z_2)z). \end{aligned} \tag{13}$$

Now for a given surface $S \in \Re^3$ the parameters u_1 and v_1 are known parameters spanning the surface S. The rest of the parameters u_2, v_2, x, y, z, x_1, y_1, z_1, x_1, y_2, z_2, can be calculated using the Equations (9) - (13). It is noteworthy that the Equations (4) - (8) are all linear in x, y, z, the Divider coordinates, while the Equations for u_2 and v_2 are highly non-linear. Furthermore, the solutions of the system must be examined to ensure that any balls with (x, y, z) as a centre and $R = \sqrt{(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2 = (x_2 - x)^2 + (y_2 - y)^2 + (z_2 - z)^2}$ as radius are indeed fully inscribed on the surface and do not contain points of it closer than their circumference.

In the next part of the paper we will try to show how we can apply the above methodology in order to compute the Divider Set for a variety of objects, in particular surfaces. It is important to highlight that the main difficulty in realising the Divider Set for a complex surface arise due to the nature of the set of the Equations (9) - (13) we are faced. In particular, one should note that the Equations (13) is non-linear and therefore in general straightforward solution techniques do not exist to solve the system of Equations (9) - (13). In what follows, we describe two methodologies namely one based on exact solutions of the Equations (9) - (13) for a certain class of surfaces and the other based on numerical solutions of the Equations (9) - (13) which can in general handle any type of parametric surface.

3 Computation of the Divider Set using Exact Solutions

As usual lets assume a surface $S \in \Re^3$ be defined by the parametric form using two parameters u and v such that $S(u,v) = (x(u,v), y(u,v), z(u,v))$. For the exact solution scheme we propose here we further assume that the surface has a special form such that,

$$x(u,v) = f_1(u)\cos(nv), \tag{14}$$

$$y(u,v) = f_2(u)\sin(nv), \tag{15}$$

where n is an integer and

$$z(u,v) = f_3(u). \tag{16}$$

Assuming our surface S can be described in the above special form, it can easily be shown that the functions associated with Equations (9) - (13) are linearly independent. Therefore, the solution method based on the Cramer's rule through the computation of appropriate Wronskian for the related matrices can be used to solve the entire system of Equations (9) - (13). Here we outline the procedures involved.

The coefficient matrix of Equations (9) - (13) considered as a linear system x, y, z as unknowns, is,

$$\begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial y_1}{\partial u_1} & \frac{\partial z_1}{\partial u_1} \\ \frac{\partial x_1}{\partial v_1} & \frac{\partial y_1}{\partial v_1} & \frac{\partial z_1}{\partial v_1} \\ \frac{\partial x_2}{\partial u_2} & \frac{\partial y_2}{\partial u_2} & \frac{\partial z_2}{\partial u_2} \\ \frac{\partial x_2}{\partial v_2} & \frac{\partial y_2}{\partial v_2} & \frac{\partial z_2}{\partial v_2} \\ 2(x_1-x_2) & 2(y_1-y_2) & 2(z_1-z_2) \end{pmatrix}. \tag{17}$$

The corresponding extended matrix, including the second part terms is,

$$\begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial y_1}{\partial u_1} & \frac{\partial z_1}{\partial u_1} & x_1\frac{\partial x_1}{\partial u_1} + y_1\frac{\partial y_1}{\partial u_1} + z_1\frac{\partial z_1}{\partial u_1} \\ \frac{\partial x_1}{\partial v_1} & \frac{\partial y_1}{\partial v_1} & \frac{\partial z_1}{\partial v_1} & x_1\frac{\partial x_1}{\partial v_1} + y_1\frac{\partial y_1}{\partial v_1} + z_1\frac{\partial z_1}{\partial v_1} \\ \frac{\partial x_2}{\partial u_2} & \frac{\partial y_2}{\partial u_2} & \frac{\partial z_2}{\partial u_2} & x_2\frac{\partial x_2}{\partial u_2} + y_2\frac{\partial y_2}{\partial u_2} + z_2\frac{\partial z_2}{\partial u_2} \\ \frac{\partial x_2}{\partial v_2} & \frac{\partial y_2}{\partial v_2} & \frac{\partial z_2}{\partial v_2} & x_2\frac{\partial x_2}{\partial v_2} + y_2\frac{\partial y_2}{\partial v_2} + z_2\frac{\partial z_2}{\partial v_2} \\ 2(x_1-x_2) & 2(y_1-y_2) & 2(z_1-z_2) & (x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2 \end{pmatrix}. \tag{18}$$

It may be assumed, without loss of generality, that Equations (9),(10),(11) are linearly independent and may be used for the solution relative to x, y and z. Here we outline the procedures involved.

Let D_x be the determinant of the matrix,

$$\begin{pmatrix} x_1\frac{\partial x_1}{\partial u_1} + y_1\frac{\partial y_1}{\partial u_1} + z_1\frac{\partial z_1}{\partial u_1} & \frac{\partial y_1}{\partial u_1} & \frac{\partial z_1}{\partial u_1} \\ x_1\frac{\partial x_1}{\partial v_1} + y_1\frac{\partial y_1}{\partial v_1} + z_1\frac{\partial z_1}{\partial v_1} & \frac{\partial y_1}{\partial v_1} & \frac{\partial z_1}{\partial v_1} \\ x_2\frac{\partial x_2}{\partial u_2} + y_2\frac{\partial y_2}{\partial u_2} + z_2\frac{\partial z_2}{\partial u_2} & \frac{\partial y_2}{\partial u_2} & \frac{\partial z_2}{\partial u_2} \end{pmatrix}. \tag{19}$$

Let D_y be the determinant of the matrix,

$$\begin{pmatrix} \frac{\partial x_1}{\partial u_1} & x_1\frac{\partial x_1}{\partial u_1} + y_1\frac{\partial y_1}{\partial u_1} + z_1\frac{\partial z_1}{\partial u_1} & \frac{\partial z_1}{\partial u_1} \\ \frac{\partial x_1}{\partial v_1} & x_1\frac{\partial x_1}{\partial v_1} + y_1\frac{\partial y_1}{\partial v_1} + z_1\frac{\partial z_1}{\partial v_1} & \frac{\partial z_1}{\partial v_1} \\ \frac{\partial x_2}{\partial u_2} & x_2\frac{\partial x_2}{\partial u_2} + y_2\frac{\partial y_2}{\partial u_2} + z_2\frac{\partial z_2}{\partial u_2} & \frac{\partial z_2}{\partial u_2} \end{pmatrix}. \tag{20}$$

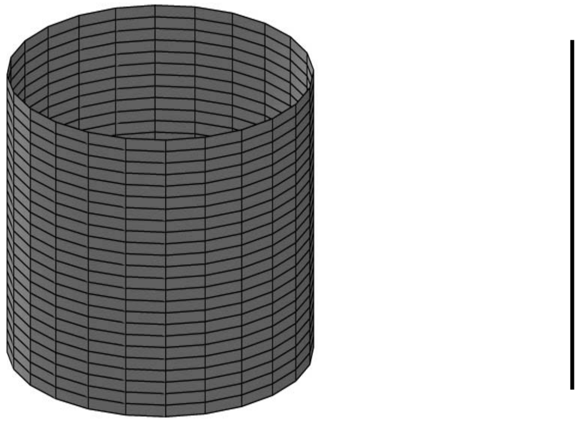
Let D_z be the determinant of the matrix,

$$\begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial y_1}{\partial u_1} & x_1\frac{\partial x_1}{\partial u_1} + y_1\frac{\partial y_1}{\partial u_1} + z_1\frac{\partial z_1}{\partial u_1} \\ \frac{\partial x_1}{\partial v_1} & \frac{\partial y_1}{\partial v_1} & x_1\frac{\partial x_1}{\partial v_1} + y_1\frac{\partial y_1}{\partial v_1} + z_1\frac{\partial z_1}{\partial v_1} \\ \frac{\partial x_2}{\partial u_2} & \frac{\partial y_2}{\partial u_2} & x_2\frac{\partial x_2}{\partial u_2} + y_2\frac{\partial y_2}{\partial u_2} + z_2\frac{\partial z_2}{\partial u_2} \end{pmatrix}. \tag{21}$$

Finally let D be the determinant of the matrix,

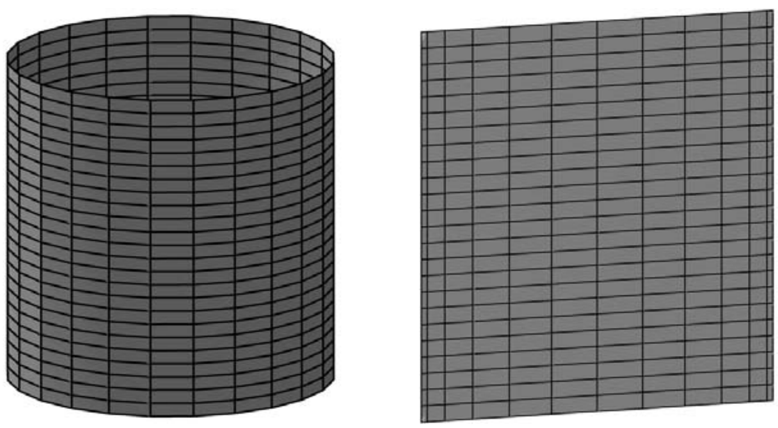
$$\begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial y_1}{\partial u_1} & \frac{\partial z_1}{\partial u_1} \\ \frac{\partial x_1}{\partial v_1} & \frac{\partial y_1}{\partial v_1} & \frac{\partial z_1}{\partial v_1} \\ \frac{\partial x_2}{\partial u_2} & \frac{\partial y_2}{\partial u_2} & \frac{\partial z_2}{\partial u_2} \end{pmatrix}. \tag{22}$$

Then Equations (9),(10),(11) yeild $x = D_x/D$, $y = D_y/D$ and $z = D_z/D$. These values, depending on u_1 and v_1 which are known and on u_2 and v_2 which are unknown, can be put into Equations (12) and (13). These Equations will now contain u_2 and v_2 as unknowns, since x, y, and z have been eliminated. Equations (12) and (13) may then be solved numerically.



(**a**) (**b**)

Fig. 1. Computation of the Divider Set of a cylindrical surface. (**a**) Surface of the cylinder represented parametrically. (**b**) The corresponding Divider Set surface, which in this case is simply a straight line through the centre of the cylinder.



(**a**) (**b**)

Fig. 2. Computation of the Divider Set of an elliptic cylindrical surface. (**a**) Surface of the elliptic cylinder represented parametrically. (**b**) The corresponding Divider Set surface, which in this case is rectangular surface through the centre of the cylinder.

3.1 Examples

In this section we show some examples where the above exact solution method can be utilised to compute the Divider Set for a number of surface shapes.

As a first example, we take a simple cylinder shape. The cylinder is parametrically represented at $x = \cos(v), y = \sin(v)$ and $z = u$. Fig. 1(a) shows the corresponding cylinder and Fig. 1 (b) shows the corresponding shape of the Divider Set which in this case is simply a line spanning along u direction through the center of the cylinder.

As a second example, we take the geometry of an elliptic cylinder. The elliptic cylinder is parametrically represented at $x = A\cos(v), y = B\sin(v)$ and $z = u$ where $A \neq B$. Fig. 2(a) shows the corresponding elliptic cylinder and Fig. 2 (b) shows the corresponding shape of the Divider Set which in this case is a rectangular section spanning along u direction passing through the centre of the cylinder.

As a third example, we take the geometry represented by the harmonic partial differential equation (PDE). The harmonic PDE we use here is the standard Laplace equation,

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) X(u,v) = 0. \tag{23}$$

Assuming the existence of periodic solutions for certain types of boundary conditions, the explicit solution of Equation (23) can be computed using separation of variables. Choosing the parametric region to be $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$ the periodic boundary conditions can be expressed as,

$$X(0,v) = P_0(v), \tag{24}$$

$$X(1,v) = P_1(v), \tag{25}$$

where the boundary conditions $P_0(v)$ and $P_1(v)$ define the edges of the surface patch at $u = 0$ and $u = 1$ respectively.

Then the explicit solution of Equation (23) can be written as,

$$X(u,v) = A_0(u) + \sum_{n=1}^{\infty}[A_n(u)\cos(nv) + B_n(u)\sin(nv)], \tag{26}$$

where

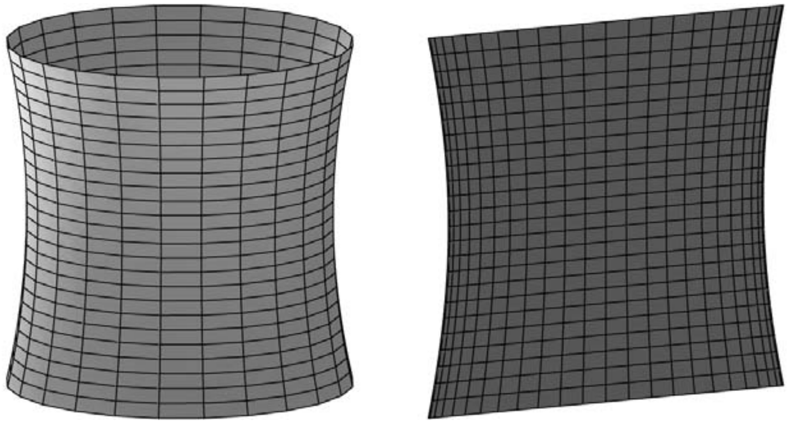
$$A_0 = a_{00} + a_{01}u, \tag{27}$$

$$A_n = a_{n1}e^{nu} + a_{n2}e^{-nu}, \tag{28}$$

$$B_n = b_{n1}e^{nu} + b_{n2}e^{-nu}, \tag{29}$$

where a_{n1}, a_{n2}, b_{n1}, and b_{n2} are vector-valued constants, whose values are determined by the imposed boundary conditions at $u = 0$ and $u = 1$.

Fig. 3(a) shows a surface generated for the Laplace equation using the solution procedure described above. Here the boundary conditions through the Solution



(**a**) (**b**)

Fig. 3. Computation of the Divider Set of a Harmonic surface. (**a**) A surface obtained by solving the Harmonic equation for a certain boundary conditions. (**b**) The corresponding Divider Set surface.

(26) can be written as,

$$x(0, v) = A\cos(v), y(0, v) = A\sin(v), z(0, v) = 0, \tag{30}$$

$$x(1, v) = B\cos(v), y(1, v) = B\sin(v), z(1, v) = 1, \tag{31}$$

where A and B are some real values. With these boundary conditions the corresponding surface is then obtained by determining the $A_0(u)$, $A_n(u)$ and $B_n(u)$ and using Solution (26). Fig. 3 (b) shows the corresponding shape of the Divider Set for the harmonic surface.

As a final example in this section we show a surface resulting to the solution from the Biharmonic equation based on the the method of surface generation proposed by Bloor and Wilson [3]. The generating equation in this case is based on the Laplace equation (23) whereby the PDE is in the form,

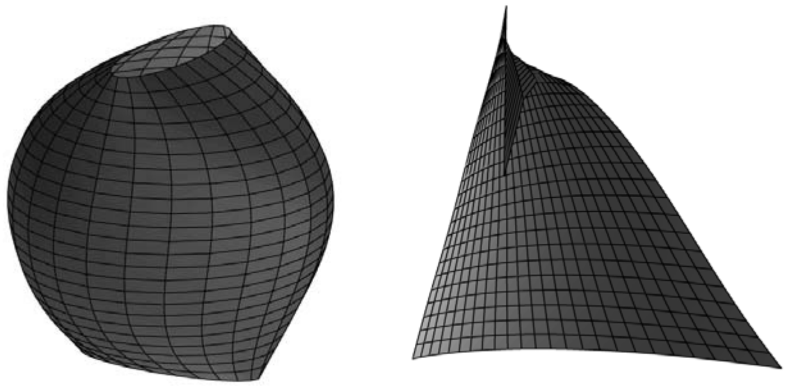
$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)^2 X(u, v) = 0. \tag{32}$$

Again, assuming the existence of periodic solutions for certain types of boundary conditions, the explicit solution of Equation (32) can be computed using separation of variables. Choosing the parametric region to be $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$ the periodic boundary conditions can now be expressed as,

$$X(0, v) = P_0(v), \tag{33}$$

$$X(1, v) = P_1(v), \tag{34}$$

$$X_u(0, v) = d_0(v), \tag{35}$$



(**a**) (**b**)

Fig. 4. Computation of the Divider Set of a Biharmonic surface. (**a**) A surface obtained by solving the Biharmonic equation for a certain boundary conditions. (**b**) The corresponding Divider Set surface.

$$X_u(1, v) = d_1(v). \tag{36}$$

where the boundary conditions $P_0(v)$ and $P_1(v)$ define the edges of the surface patch at $u = 0$ and $u = 1$ and the boundary conditions $d_0(v)$ and $d_1(v)$ define the derivatives at the edges of the surface patch at $u = 0$ and $u = 1$.

The explicit solution of the Biharmonic Equation (32) is also given by the Solution (26) where the vectors $A_0(u)$, $A_n(u)$ and $B_n(u)$ comprise of added terms to accommodate the higher order of the equation. Detailed description of this solution scheme can be found in [3].

Fig. 4(a) shows a surface generated through the Biharmonic equation using the solution procedure described above. Here the boundary conditions for the surface can be written as,

$$x(0, v) = A\cos(v), y(0, v) = A\sin(v), z(0, v) = 0, \tag{37}$$

$$x(1, v) = B\cos(v), y(1, v) = B\sin(v), z(1, v) = 1, \tag{38}$$

$$x_u(0, v) = C\cos(v), y_(0, v) = C\sin(v), z_u(0, v) = 0, \tag{39}$$

$$x_u(1, v) = D\cos(v), y_u(1, v) = D\sin(v), z_u(1, v) = 0, \tag{40}$$

where A, B, C, D are some real values. With these boundary conditions the corresponding surface is then obtained by determining the $A_0(u)$, $A_n(u)$ and $B_n(u)$ and using an extended version of the Solution (26). Fig. 4(b) shows the corresponding shape of the Divider Set for the Biharmonic surface.

4 Computation of the Divider Set using Numerical Solutions

In the above section we have described how the Divider Set for a certain class of parametric surfaces can be computed through an exact solution method for the system of equations involved. In this section we show how the Divider can be computed to general parametric surfaces through the use of a numerical solution procedure.

As we have highlighted earlier, the system of equations, evaluated over the relations involving the surface in question becomes a nonlinear system of algebraic equations with unknowns quantities. To solve this in general we proceed as follows. First, we observe that the equations (9) - (13), form a system of linear equations, with respect to the variables x, y, and z. We note that this system is solvable if the ranks of the coefficient matrix and the augmented matrix are equal. Using the Gauss elimination method we can, therefore, reduce those matrices to echelon forms. Then we can state that, the equality in the ranks of the matrices implies that a set of two equations of the unknowns (u_2, v_2) must be valid.

Another point to note is that the system of equations involve nonlinear algebraic polynomials. To solve it we work numerically by using the Newton-Raphson method [10]. Sufficient initial conditions for this method are obtained by means a deepest descent algorithm [21]. Substituting these results backwards and solving the linear systems, we obtain the values of x, y and z.

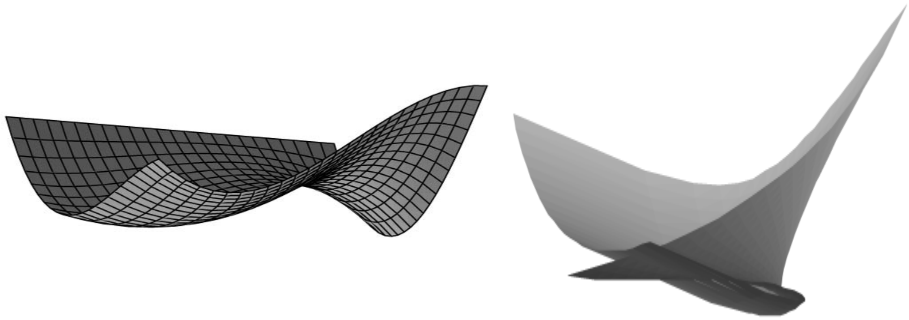
The above procedure is undertaken over the discrete domain containing the points (u_1, v_1). Thus, the whole procedure is repeated for each discrete point (u_1, v_1), which results in a discrete set of points of the Divider Set.

4.1 Examples

In this section we show examples where we have utilised the general numerical procedure outlined above in order to compute the Divider Set.

As a first example we take a polynomial surface based on the well known Bézier formulation. The particular form of the Bézier surface is given by the polynomial equations,

$$\begin{aligned} x(u,v) = 3v + 9uv - 18u^2v + 9u^3v - 18uv^2 + 9uv^3 + \\ 36u^2v^2 - 18u^2v^3 - 18u^3v^2 + 9u^3v^3, \end{aligned} \tag{41}$$



(**a**) (**b**)

Fig. 5. Numerical computation of the Divider Set for a polynomial surface. (**a**) The polynomial surface obtained through Bézier formulation. (**b**) The corresponding Divider Set surface.

$$y(u,v) = 1 + 3v - 9v^2 + 5v^3 + 3u^2 - 4u^3 - 9u^2v + 6u^3v + 3uv^3 + 9u^2v^2 - 3u^2v^3 - 5u^3v^3, \tag{42}$$

$$z(u,v) = 3u + v. \tag{43}$$

Fig. 5(a) shows the a polynomial surface obtained through Bézier the formulation. Fig. 5(b) shows the corresponding shape of the Divider Set obtained through the numerical procedure described above. Note that the Divider Set in this case was obtained as a discrete set of surface points upon which a separate triangulation procedure has been adopted to the discrete points in order to create a surface for visualisation purposes.

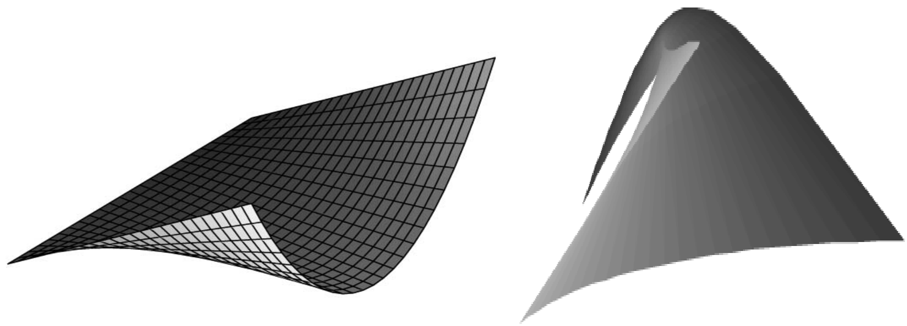
As a second example we take a Biharmonic surface based on the Equation (32) where the necessary boundary conditions are based on the following polynomial equations,

$$x(u,v) = 3v, \tag{44}$$

$$y(u,v) = 3v - 9uv - 9u^2v, \tag{45}$$

$$z(u,v) = 3u. \tag{46}$$

Thus the necessary boundary conditions $X(0,v)$, $X(1,v)$ are taken at the $u = 0$ and $u = 1$ from the polynomial equations (44) - (46) while the boundary conditons $X_u(0,v)$, $X_u(1,v)$ are all taken to be zero. With these boundary conditions, the four sided surface patch shown in Fig. 6(a) is obtained by solving the Equation (32) using a pseudo-analytic solution scheme presented in [4]. Fig. 6(b) shows the corresponding shape of the Divider Set obtained through the numerical procedure described above. Again note that the Divider



(**a**) (**b**)

Fig. 6. Numerical computation of the Divider Set for a Biharmonic surface. (**a**) A Biharmonic surface obtained. (**b**) The corresponding Divider Set surface.

in this case was obtained as a discrete set of points upon which a separate triangulation procedure has been adopted to the discrete points in order to create a surface for visualisation purposes.

5 Conclusions

In this paper we have described a concept for classification of complex geometry based on the concept of the Divider Set. It is a novel alternative concept to maximal disks, Voronoi sets and cut loci, based on a formal definition relating to topology and differential geometry. The emphasis of this paper has been to introduce the concept of the Divider Set within the context of definition morphology of objects and its computation for complex 3-dimentional geometry. In particular in this paper we have shown how the Divider Set can computer for surfaces described in parametric form. Two forms of solutions has been described, one analytic which takes advantage of the special parametric form of the surface and the other a numerical solution which can be utilised for general parametric surfaces.

There a number of extensions to this work. In this paper, we have described the methods of computing the Divider Set for parametric surfaces. While parametric surfaces are applicable to a number of areas there are situations where surfaces or solids cannot be defined in parametric form. In which the solution methods we have discussed would need further developments. Another area to extend this work is to study the Divider Set for discrete cases. This has applications in situations where the geometry and images in question are available in discrete form. Our ongoing and future work in this topic aims to extend our research efforts in these areas.

References

[1] C. Arcelli and G. Sanniti di Baja. Ridge Points in Euclidean Distance Maps. Pattern Recognision Letters, **13**(4) (1992) 237-243

[2] Bakopoulos, Y., and Stavrinos, P.C: The Generalized Curvature of Locally Convex Type. ELEFTERIA (S.P. Zervos ed.), **4B** (1986) 267-280

[3] M. I. G. Bloor and M. J. Wilson. Using Partial Differential Equations to Generate Freeform Surfaces. Computer-Aided Design, **22** (1990) 202-212

[4] M. I. G. Bloor, and M. J. Wilson. it An analytic Pseudo-Spectral Method to Generate a Regular 4-sided PDE surface Patch. Computer Aided Geometric Design, **22**(3), (2005) 203-219

[5] H. Blum. A transformation for extracting new descriptions of shape. In W. Wathen-Dunn, editor, Models for the Perception of speech and Visual Form, pages 362-380. MIT Press, (1967).

[6] Boissonnat, J. D.: Geometric Surfaces for 3-Dimensional Shape Representation. ACM Transactions on Graphics, **3**(4) (1984) 244-265

[7] H. I. Choi, S. W. Choi and H. P. Moon. Mathematical Theory of Medial Axis Transform, Pacific Journal of Mathematics, **181**(1) (1997) 57-88

[8] R. Duda and P. Hart. Pattern Classification and Scene Analysis. Wiley-Interscience, New York, (1973).

[9] D. S. Fritsch, S. M. Pizer, B. S. Morse, D. H. Eberly and A. Liu. The Multiscale Medial Axis and Its Applications in Image Registration. Pattern Recognition Letters, **15**(5), 445-452, (1994).

[10] N. Kalouptsidis, Signal Processing Systems, Wiley-Intersience, (1997).

[11] L. Lam, S.-W. Lee, and C. Suen. Thinning Methodologies - A Comprehensive Survey. IEEE Transactions on Pattern Analysis and Machine Intelligence, **14**, 869-885, (1992).

[12] F. Leymarie, and M. D. Levine. Skeleton from Snakes, Progress in Image Analysis and Processing, World Scientific Singapore (1990)

[13] L. R. Nackman and S. M. Pizer. Three-Dimensional Shape Description using Symmetric Axis Transform. IEEE Transactions on Pattern Analysis and Machine Intelligence, **7**(2) (1985) 187-202

[14] R. Ogniewicz. Discrete Voronoi Skeletons. Hartung-Gorre Verlag, Konstanz, Germany, (1993)

[15] J. O'Rourke. Computational Geometry in C. Cambridge University Press, New York, (1994)

[16] N. M. Patrikalakis and H. N. Gursoy. Shape Interrogation by Medial Axis Transform. In: Ravani, B. (ed.): Advances in Design Automation: Computer-Aided Computational Design, ASME **1** (1990) 77-88

[17] T. Pavlidis. Algorithms for Graphics and Image Processing. Computer Science Press, Rockville MD, (1982)

[18] S. J. Ruuth and B. Merriman, Convolution-Generated Motion and Generalized Huygens' Principles for Interface Motion, SIAM Journal on Applied Mathematics **60**(3) (2000) 868 - 890

[19] E. C. Sherbrooke, N. M. Patrikalakis and F. Wolter, Differential and Topological Properties of Medial Axis Transforms, Graphical Models and Image Processing **58**(6) (1996) 574-592

[20] Y. F. Tsao and K. S. Fu. A Parallel Thinning Algorithm for 3-D Pictures. Computer Graphics Image Process, **17** (1981) 315-331

[21] G. R. Walsh, Methods of Optimization, Wiley-Intersience, (1977)