Arithmetic Fractal Structures and Number Theoretic Properties in the measurement protocols for the CHSH and Leggett-Garg inequalities

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Abstract: We present numerical evidence on the fractal structure of the combinatoric powerset restricting all binary words corresponding to the set of any finite length photon measurements and their coincidence rates for both the Clauser-Horn and the Leggett-Garg proposed types of inequalities associated with Bell type violations. This structure originates in purely number theoretic properties of combinatoric sets and represents a logical restriction independently of the underlying physical process. A set of MATLAB codes for producing images and movies of said structures in higher dimensional parametric space is provided in an appendix.

1. Introduction

Lately, some important results appeared in the literature on the issue of loophole-free detection of Bell inequality violation by Christensen [1] and Hensen [2] which are considered the tombstone for all local hidden variable theories. As there is an endless discussion on whether these inequalities really exclude any hidden variables or whether there are non-local such variable models, we chose to examine a particular property associated with the type of variables actually been measured. This property is not due to the physics behind the measurements but is strictly associated with the Boolean variables been used and it is thus an objective property due to purely mathematical reasons that sets certain restrictions on the set of all possible such measurements and which have not appeared elsewhere in the literature known to us.

In a previous report [3], we have examined the generic structure of combinatoric powersets of symbolic words of any n-ary alphabet and showed that they are all associated with a particular type of a self-affine map producing the next level of a combinatorial hierarchy. The only other example of the study of combinatorial hierarchies for foundational issues in physical theories seems to have been provided by previous studies of Pierre Noyes in the 70s and 80s [4], [5]. There are certain similarities and differences between the two approaches the main one being that Noyes mainly focuses on the notion of certain Abelian group structures over 2^N binary objects called *"Discriminatory Sets"* while our approach aims to reveal a generic discrete self similar structure associated with certain underlying recursive and thus algorithmic relations over any *n*-ary objects or "words" which are also associated with the polynomial representations of the integers.

In what follows we first examine the particular case of a toy model for experiments like those performed by Christiansen and Hansen. Our strategy is to derive first a global function as a functional composition of the underlying supposedly continuous activity and a set of Boolean filters the main observation been that all actually measured quantities are binary variables and as such they form a discrete lattice over the supposedly continuous background. At this point it is imperative to stress that from a purely empirical point of view providing a so called compact support to all such variables is an abstract axiom beyond verification necessary only to ease calculations.

For all such filtered discrete sets of measured variables one can always provide a global map over the set of all possible combinations. Using previous knowledge on the perfectly ordered and in fact recursive structure of such sets we proceed to analyze the degree at which recursivity and self similarity are indeed inherited by the application of the particular type of functional forms of both the CHSH and the Leggett-Garg formulas used by the experimentalist. For as long as any such test can be directly performed via computer simulation, the results are of global validity independently of any type of underlying randomness. This can then be interpreted as saying that an actual experiment is equivalent to a particular random sampling of a preexisting discrete fractal structure. This not recognized generic characteristic will then influence the resulting statistics.

2. Numerical examples in CH protocol

The basic structure of a physical model for getting results as these by Christensen and Hansen has the typical structure of a composite function $(g_i \circ f)$ where f is a 4-fold degenerate map sending each of the 4 equal subintervals of [0, 2π] to [0, 1] under A and B detectors as

(1) $f_{A,B} = \cos(A - \vartheta_{A,B}) + noise$

In (1) g_i is one of a set of Boolean filters which can be written with the aid of the auxiliary variables

$$(2)\ u = (f > threshold), v = ((1 - f) > threshold)$$

in the form

$$(3)\ \begin{aligned} g_{doubles} &= ((u \wedge v) \vee (u' \wedge v')) \\ g_{misses} &= ((\sim u \wedge \sim v) \vee (\sim u' \wedge \sim v')) \\ g_{anticorr} &= ((u \wedge v') \vee (v \wedge u')) \end{aligned}$$

In the above, primed variables correspond to a different threshold A or B. One can then obtain a global map as follows. For all values in the interval [0, 2π] and for all values of the two thresholds one has the set of characteristic functions over the reals as $g_i(f(x), y, z), x \in [0, 2\pi], y, z \in [0,1]$. One can then break this decomposition in two parts by first extracting the form of *u* and *v* as matrices and rewrite the above in the form $g_i(u, v, u', v'), u, v \in \{0,1\}$. Any possible measurement can then be seen as a random walk over the resulting logical matrices which are generic for any random measurement. Moreover, a collection of N measurements can be interpreted as a superposition of N such matrices in which case one can apply the polynomial representation of the integers and identify the superposition of each entry G_{ij} as a single integer of the form $g_i(N) = \sum_{i=0}^{N-1} 2^i G_{ij}$. Hence, any result concerning additive variables like the single and double counters becomes mathematically identical with an arbitrary random permutation of the so called Digit-Sum [Wolfram] function over the set of the integers in successive intervals $[0, ..., 2^N - 1]$. We have already shown in [3] that the above sequence when computed over the integers exhibits a clear recursive structure (discrete self-similarity) due to the particular way new bits are added over exponential intervals. As a result, one can derive the following recursion

$$(4)\ \begin{aligned} DS_{n+1} &\leftarrow \{DS_n, DS_n + 1\} \\ DS_0 &= \{0\} \end{aligned}$$

In fact, the recursion in (4) can actually be solved with the introduction of special "pulse" functions y_P as follows

$$(5)\quad DS(n)=\sum_{k=1}^{2^N} y_P\left(n,2^k,2^k\right)$$

In (5), the pulse function $y(n,\tau_1,\tau_2)$ is a periodic repetition of binary blocks $\left\{\{0\}^{\tau_1}\{1\}^{\tau_2}\right\}$ with a total period $P=\tau_1+\tau_2$.

As an example of constructing such matrices we may use a simple vectorized octave/matlab snippet

```
x = 0:step:2*pi;
y1 = cos(x).^2; y2 = 1 - y1;
step = 1/length(y1);
for x0 = 0:step:1-step
   f = (y1 > x0);     g = (y2 > x0);
   and1 = f.*g;
   xor2 = abs(f - g);
   sav1 = [sav1, and1];
   sav2 = [sav2, xor2];
   Amat = [Amat; and1];
   Xmat = [Xmat; xor2];
end
```

The resulting matrices are shown in figures 2 and 3. An example of a double detection matrix can be made with the below additional snippet only for a cross section of equal thresholds in the full 4-dim space

```
mat = [];
for i=1:floor(length(sav1)/2)
   for j=1:floor(length(sav1)/2)
      mat(i, j) = or(sav1(i), sav2(j));
   end
end
```

This results in the fractal structure of which a portion is shown in figure 4 due to memory restrictions.

The above are incomplete in that the full 4-dim global map would require a large succession of images either as an unfolding over a planar map or as an exponentially large movie. Instead, one may use a different strategy. Following the generic recipe used for such measurements one can construct a global map by taking all possible binary strings corresponding to an observation depth N and form the Cartesian product for many different variables. This is a well ordered array, which also happens to be an arithmetic fractal set as shown in figure 1 due to the existence of a simple self-affine transform of each row to the next, a fact not stressed enough in standard textbooks. People in Digital Design tend to work row-wise (with particular words) where this fact is overlooked due to the rows exhibiting quasi-periodicity being isomorphic with the output of binary counters or square oscillators with exponentially increasing periods.

3. Recursivity of the set of all measurements

As a result, any finite size measurements necessarily belong into a self-similar extension of the above in an arbitrary exponential interval. We then apply simultaneously the CHSH formula to all of the above string combinations to produce another ordered set of outputs. This always results in another self-similar structure. A code that produces a succession of images and a movie is given in the Appendix. Another code checking the structure of CHSH for the whole combinatorial superset is also given there next.

One can in principle apply the same strategy for the case of the so called Leggett – Garg inequality which can be made slightly easier by making use of only three variables entering the global map. This one in particular results in structures similar to a Sierpinsky's gasket that can be seen by making a movie (3rd code in App.) The above shows that whatever the statistics, there is at least a part which is 'objective' as it has been independently constructed out of endomorphisms from the integers into their polynomial representations.

For both cases, the overall superset results is characterized by a recursive structure inherited by the simple self similar nature of the basis set of all binary words for any closed interval $[0,\ldots, 2^N]$ and for any arbitrary length N as the one in figure 1. Recursive relations can be found in such cases with specialized algorithms which most often are generalizations of the primitive recursion given in (4). To ease the understanding of the above statement one may notice that for all countable discrete n-dimensional sets there always exist an equivalent one dimensional representation without loss of information. Taking the Leggett-Garg case as an example one may also write

$$(6)\ f_{i,j,k} = \left\langle \sum_{i\neq j, i=1, j=1}^{3} Q_i Q_j \right\rangle \cong f_l, \quad l = i + 2^N j + 2^{2N} k$$

The above stands for an "unfolding" of the original bounded 3-dimensional set such that self similarity will exhibit itself through the existence of a set of appropriate functions $\{g_i(x)\}_{i=1}^{M}$ such that for any property $|P|$ like the ordinary average there will be a generic recursive formula or "program" of the form

$$(7)\ \begin{array}{l} |P|_{n+1} \leftarrow \{|P|_n, g_1(|P|_n), g_2(|P|_n), \ldots g_M(|P|_n)\} \\ |P|_0 = \{x_0\} \end{array}$$

We may then say that the superset of all possible measurements is computable from a primitive initial seed. A natural question then is how much is this property distorted in the case of non-classical violations.

For this reason, we first recognize that any deviation from classical behavior must be due to the presence of some phase factor that essentially moves the original variables away from their classical "*dichotomic*" values which gives them their Boolean character. One may then simulate this effect in our third code with the introduction of an exemplary $\cos(\varphi)$ term which in the worst of cases could come from a random sampling of the unit circle. In practical applications the phase factor φ will be some complicated function of space and time depending on the details of the associated wavefunction. We may then rephrase the original in the form

$$(8)\ f_{i,j,k} = \left\langle \sum_{i\neq j, i=1, j=1}^{3} Q_i Q_j \cos\varphi_{i,j} \right\rangle$$

We then ask whether the product space $\mathbf{W}^3 \times [S^1]_{\delta S}$ where $[S^1]_{\delta S}$ stands for a random sampling of the circle of minimal arc δS and the set of all binary words W for all integers in $[0,\ldots,2^N]$. In fact, one may see in the above a kind of discrete fibration around the unit circle by taking the one dimensional correspondence of indices again and the associated words via the polynomial representation as in (6) and their mapping on a rational interval which could again be the [0, 1] or $[S^0]$.

In any such case, randomization by an arbitrary sampling can at most produce a random fractal or multi-fractal structure but one can also simple expand the parametric space from a 3-dimensional to a 6-dimensional space including also

$\{\varphi_{i,j}\}$ as independent variables. Given that any such finite accuracy sampling of the unit circle can be put into correspondence with the integers one regains the overall self similar structure and thus the recursivity even in the case of the superset of all such non-classical measurements. Using the third code in the appendix with the *r* flag non-zero will introduce random perturbations. Any other formula for phase can be checked but changing to a 6-loop and given enough memory one can see at least several cross sections of the resulting 6 dimensional hypercube.

The overall result is that in all such cases, the recursivity of the total combinatorial powerset of any possible measurements of finite accuracy is *persistent.* Whether this reflects some deeper property that nature has actually made use of or has been restricted by is questionable but we have sufficient evidence for all such constructs being recursive and thus hiding a kind of potentially universal algorithmic structure.

4. Discussion and Conclusions

The existence of global self similar structures for discrete maps like those presented in the case of the CHSH and Leggett-Garg inequalities is an unavoidable consequence of the mathematical structure of such object as the so called factorial and block designs [8] in general. The significance of this observation permeates many other fields and it will be the subject of a subsequent article. An important ingredient of such constructs is that different correlations between symbolic sequences or words in n-ary alphabet and their block structure are preserved by the non-commuting character of the simple concatenation map between strings.

To give another example of the generality of the above one may examine the expansion of all multinomial product-sums of a non-commuting multiplication given as $(\mathrm{A}_1 + \mathrm{A}_2 + ... + \mathrm{A}_v)^N$ where $\lfloor A_i, A_j \rfloor \neq 0$. The same can also be said about computational structures on a functional setting where multiplication is replaced with functional composition. There is a particular analogy between such constructs and the interpretation of Feynman integrals as running over the set of all possible paths. The significance of the above will be reported elsewhere.

The case of the particular functions used in both the CHSH or Leggett-Garg inequalities is not unique. In principle any other bitwise functions or relations defined similarly on a combinatorial powerset will more or less inherit the internal self affine nature of those sets for any n-ary alphabet although at the moment it is hard to find an exact quantitative index for the degree of such inheritance. As this property is generic and also persistent in both classical and quantum cases the author believes it deserves further investigation especially in the case of averages over very large sets.

References

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Appendix: MATLAB Codes

```
function bellavg(n)
% study of the Bell averages over their combinatoric set
% For each pair of polarization orientations there exist a
% symmetric function which depends on a 3-bit word
% 1st path bit, 2nd path bit and 3rd coincidence bit.
clc, close all
dim = 2^n;
lex = fliplr( ff2n(n) ); %lexicon of possible observations set
for i=1:dim
   s1 = lex(i, :); % 1st path bits of n consectuive detections
   for j=1:dim
      s2 = lex(j, :); %2nd path bit
      for k=1:dim
         s3 = lex(k, :); %3rd coincidence bit
         % extract the 4 coincidence functions N++, N+-, N-+, N--
         N00 =0; N01 = 0;
         N10 = 0; N11 = 0;
         for m=1:n
            if s1(m) ==0 && s2(m) ==0 N00 = N00 + s3(m);end
            if s1(m) ==0 && s2(m) ==1 N01 = N01 + s3(m);end
            if s1(m) ==1 && s2(m) ==0 N10 = N10 + s3(m);end
            if s1(m) ==0 && s2(m) ==1 N11 = N11 + s3(m);end
         end
         r1 = N00 + N11; r2 = N01 + N10;
         w(i, j, k) = (r1 - r2)/(r1 + r2);
      end
   end
end
moviename = 'Bell.avi';
mov = avifile(moviename);
for k=1:dim
    imagesc(w(:, :, k))%, colormap(gray),
    title(['mean density/frame : ',num2str( 100*mean(mean(w(:,:,k))) )]), pause(0.1)
    F=getframe(gca);
    mov=addframe(mov,F);
end
mov=close(mov);
end
```

```
function chshcube(n, samples)
% study of the CHSH inequality over its full combinatoric set.
% For each pair of polarization orientations there exist a
% symmetric function which depends on a 3-bit word
% 1st path bit, 2nd path bit and 3rd coincidence bit.
% First compute all instances of Bell's average, then extract
% the fractal subset satisfying the chsh inequality
clc, close all
dim = 2^n;
lex = fliplr( ff2n(n) ); %lexicon of possible observations set
for i=1:dim
   s1 = lex(i, :); % 1st path bits of n consectuive detections
   for j=1:dim
      s2 = lex(j, :); %2nd path bit
      for k=1:dim
         s3 = lex(k, :); %3rd coincidence bit
         % extract the 4 coincidence functions N++, N+-, N-+, N--
         N00 =0; N01 = 0;
         N10 = 0; N11 = 0;
         for m=1:n
            if s1(m) ==0 && s2(m) ==0 N00 = N00 + s3(m);end
            if s1(m) ==0 && s2(m) ==1 N01 = N01 + s3(m);end
            if s1(m) ==1 && s2(m) ==0 N10 = N10 + s3(m);end
            if s1(m) ==0 && s2(m) ==1 N11 = N11 + s3(m);end
         end
         r1 = N00 + N11; r2 = N01 + N10;
         w0(i, j, k) = (r1 - r2)/(r1 + r2);
      end
   end
end
wmax = max(max(max(w0)));
wmin = min(min(min(w0)));
disp(['Min.: ',num2str(wmin),'Max.: ',num2str(wmax)])
% Random sampling over the whole cube will give a shadow of the
% boundary between values below and above the [-2, 2] values.
sumv = 0;
sg = [1, -1, 1, 1];
w = zeros(dim,dim,dim);
for i=1:samples
   chsh = 0;
   xyz = randint(3, 4, [1, dim]);
   for j=1:4
      i0 = xyz(1, j); j0 = xyz(2, j); k0 = xyz(3, j);
      chsh = chsh + sg(j)*w0(i0, j0, k0);
```

```
    end
    if abs(chsh) <= 2, sumv = sumv + 1; w(i0, j0, k0) = 1; end
end
disp([num2str(sumv),' violations over a volume of ', num2str(dim^3)])
disp([num2str(100*sumv/dim^3),' % total'])
end
```

```
function checklg(n, r, moviename, sideview)
% study of the Legget-Garg bound
% n <- observation vectors length
% r <- random phase
% moviename <- avi file
% sideview <- cross section choice for movie
clc, close all
dim = 2^n;
lex = fliplr( ff2n(n) ); %lexicon of possible observations set
for i=1:dim
     s1 = 1 - 2*lex(i, :); % change encoding to +/- 1
    for j=1:dim
        s2 = 1 - 2*lex(j, :);
        for k=1:dim
            s3 = 1 - 2*lex(k, :);
            if r % randomize average
                r1 = rand(1,n); r2 = rand(1,n); r3 = rand(1,n);
                v = (s1.*s2)*r1 + (s2.*s3)*r2 - (s1.*s3)*r3; %Randomized
Leggett - Garg terms
            else
                v = s1.*s2 + s2.*s3 - s1.*s3; %Unperturbed Leggett - Garg
terms
            end
            w(i, j, k) = sum(v)/n;
         end
    end
end
d = mean( w(:) );
disp(['mean density ', num2str(100*d),'%'])
moviename = [moviename,'.avi'];
mov = avifile(moviename);
for k=1:dim
    switch sideview
        case 1, imagesc(w(k, :, :)), pause(0.1)
        case 2, imagesc(w(:, k, :)), pause(0.1)
        case 3, imagesc(w(:, :, k)), pause(0.1)
        otherwise, error('arg 4 < 1 or > 3 not allowed!')
    end
     F=getframe(gca);
     mov=addframe(mov,F);
end
mov=close(mov);
end
```

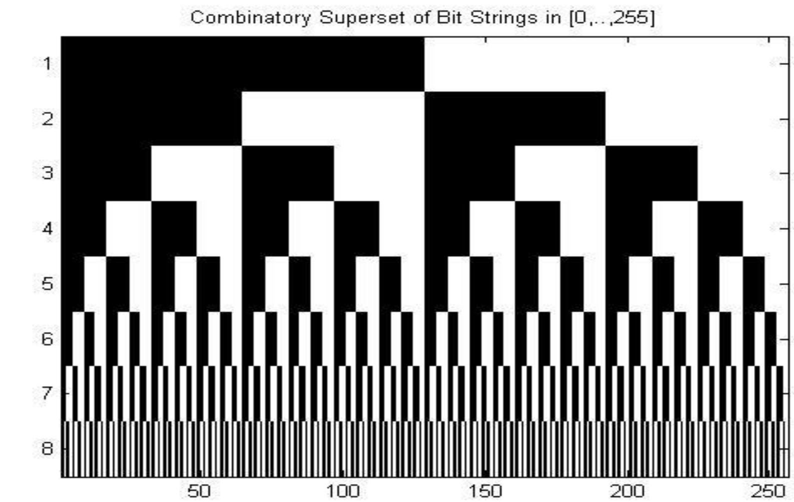

Fig. 1 (Black → 0, White → 1)

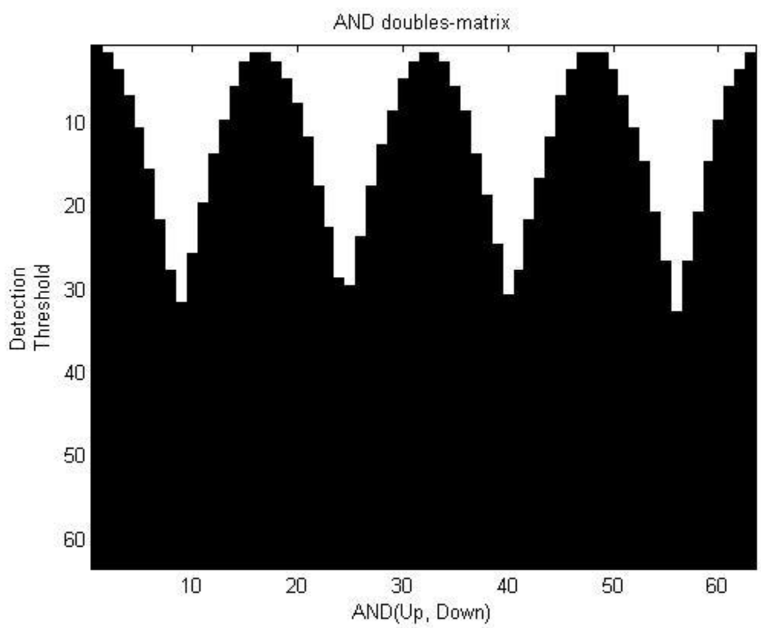

Fig. 2

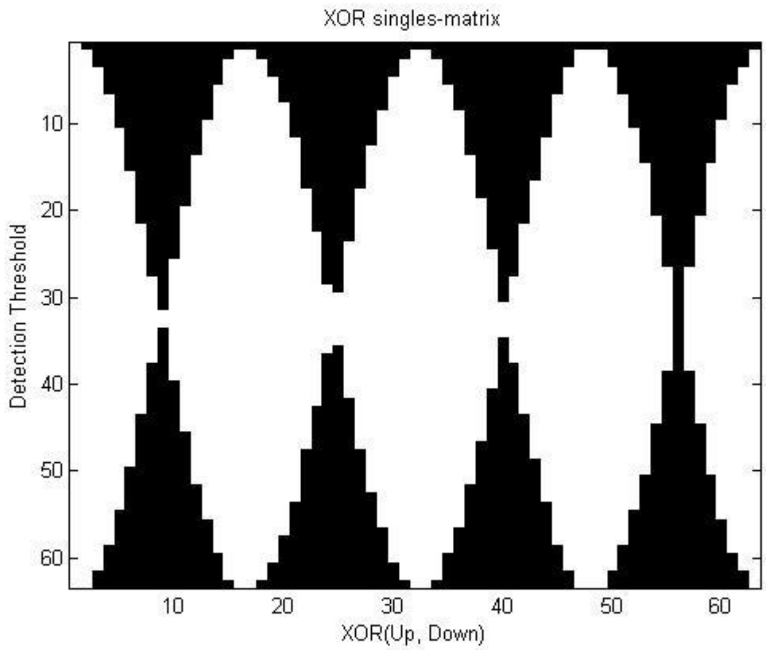
XOR singles-matrix
Detection Threshold
10
20
30
40
50
60
10
20
30
40
50
60
XOR(Up, Down)

Fig. 3

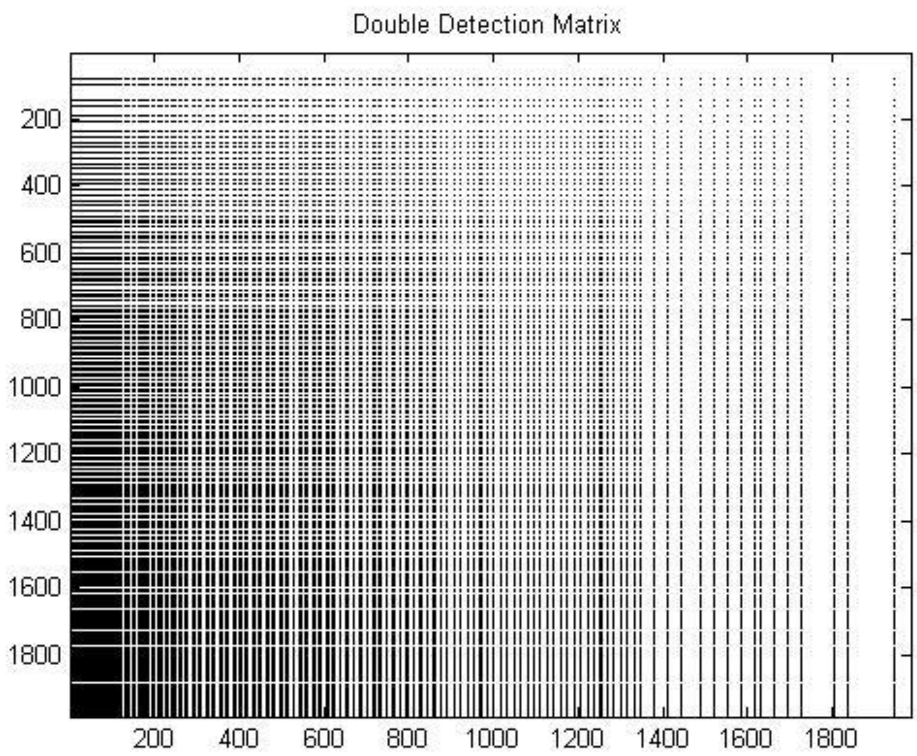
Double Detection Matrix
200
400
600
800
1000
1200
1400
1600
1800
200
400
600
800
1000
1200
1400
1600
1800

Fig. 4