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Part II : *Beltrami Field Helicity Modulators*

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A NEW GENERIC CLASS OF BELTRAMI "FORCE-FREE" FIELDS PART I : Beltrami Field Helicity Modulators

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A New Class of Beltrami "*Force-Free*" Fields
Part II : *Beltrami Field Helicity Modulators*

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Following a previous report, we present a new class of instruments capable of modulating their interior electromagnetic field so that we can locally approximate vacuum solutions of Force-Free Fields. This result is utilized to create a new type of modulators which we have termed Helicity Modulators of which the type of propagation differs from ordinary polarization.

1. Introduction

In a previous report [1] we presented a general methodology of geometrical nature for obtaining solutions of the Beltrami problem. We also considered the case of both spatial and temporal modulation of the helicity content of such solutions which may vary locally. In this second part we present a special category of devices based on a particular formation of magnetic boundary conditions to shape the magnetic field in the form of a Beltrami field which are then to be excited by an additional time-dependent excitation. We have already proven in [1] that any temporal variation of a Beltrami-like magnetic component will result into a similar electric component with generally different eigen-vorticities. To this purpose we use a special arrangement of magnetization current known as a *Halbach Array* [2].

At 1988, K. Halbach introduced a class of magnetic cylinders with special magnetization for use in high energy physics. Later on, these became widely popular under the name of *Halbach Arrays* and their applicability spread widely to other engineering sectors like brushless motors. In a number of papers, Leupold and Potenziani [11], [12] and later Coey et al. [14] showed how to create a variety of unusual multipolar configurations of magnetic field.

In the following, we examine the subject of field configurations in such magnetic systems which may also include constant or varying current sources at the boundaries. Specifically, we present the steps that lead to the gradual specification of the elements of a class of instruments that should be able to modify the helicity of the interior magnetic flux to conform with a solution of (1) and (2) at least locally. We refer to the first part of this report [1] for details on the subject of Force-*Free Fields* also known in the mathematical and hydrodynamics literature as the more general *Beltrami* fields. In section 2 we propose a methodology for setting up the necessary boundary conditions with cylindrical symmetry for the formation of an approximate Beltrami-like magnetic field in their interior, while in section 3 we expand our previous approach in spherically symmetric systems also known as *Halbach Spheres.* In section 4 we examine the magnetic forces appearing in the boundaries of the systems proposed. In section 5 we treat the subject of radiation from such instruments and we attempt to fit some of the already known solutions for Force-Free Fields in the case of the vacuum fields in the intermediate region between the boundaries of devices that we have termed *Beltrami Field Helicity Modulators* (BFHM). In section 6, we attempt to derive a general methodology for building Beltrami flows from arbitrary vacuum solutions of Maxwell equations. In section 7 we note some peculiarities of the particular class of fields and we conclude on the possibility of a new form of engineering electromagnetism that may lead to new communication protocols.

2. Beltrami Field Modulators: Cylindrical Symmetry

A Beltrami field is one that satisfies the equations

$$\nabla \times B = \lambda(\mathbf{r}, t) B \qquad (1)$$

$$\mathbf{B}\nabla\lambda(\mathbf{r},t)=0,\nabla B=0 \quad (2)$$

The scalar function $\lambda(\mathbf{r})$ is the *eigen-vorticity,* which is a generalization of the eigenvalue of the rotation operator. Such fields have been initially introduced in hydrodynamics for the classification of vortex-like structures. A brief but comprehensive review of the issues involved as well as of the relevant literature has been given in Part I of this report [1]. Several special solutions can be found in the literature with applications in Solar Physics, Astrophysics, Atmospheric Physics and Magneto-hydrodynamics. For the cylindrical case there exists a nice formula known as the *Lundquist* solution [2] given by $B=B_0[0,J_1(\lambda r),J_0(\lambda r)]$. This has the advantage that it can exist in a bounded domain with an external magnetic field $B_{\partial D}=(\lambda B_0/r)[0,J_1(\lambda r),0]$. On the other hand our strategy is based on the solution of an inverse problem. Given that we can locate an appropriate class of solutions, the individual dipole elements for the surface magnetizations of the necessary boundaries have to be found. In order to completely solve the magnetostatic problem we have to find an equivalent current source through $\nabla\times B$ and then take the Helmholtz analysis the resulting current into solenoidal and rotational components. The second part is ot be interpreted as the equivalent magnetization.

We start from the fundamental construction of a magnetic cylinder based on Halbach rings. Each Halbach ring [] has a magnetization given by

$$M=M_r(\sin(k\theta)\hat{r}-\cos(k\theta)\hat{\theta}) \quad (3)$$

The most important property of such a device is its capability to concentrate the overall magnetic flux in the interior or the exterior of the device according to the sign of the *k* coefficient. For $k=2$ one gets a constant field in the interior while higher multipoles appear for $k>2$. For *k* negative, it is also possible to invert the flux towards the interior by a simple reversal of the individual dipole moments.

The purpose of our construction is to modulate the flux lines in each section of the cylinder into a helical shape characterizing Beltrami flows, and modulate it by simple vectorial addition of an appropriate temporally varying field $(0,0,H_z(t))$ or by a rotation $\omega(t)$ either of an exterior or an interior modified Halbach cylinder. In Fig. 1 we show the cross-section of a fundamental element known as a cyclic Halbach array. In order to extend their influence towards the z axis we also introduce a varying magnetization $M_r(z)$ and we increase the symmetry of the interior field putting a similar inner cylinder with the same but negative value of *k*.

In order to fit the components of the total interior field into a solution of (1) and (2), we write explicitly the equations for each component and search for appropriate functions of the coefficients satisfying

$$-\partial_z H_\varphi=\lambda H_r \quad (4\text{-}1)$$

$$\partial_z H_r=\lambda H_\varphi \quad (4\text{-}2)$$

$$\partial_r(rH_\theta)-\partial_\theta H_r=0 \quad (4\text{-}3)$$

$$\partial_r(rH_r)-\partial_\theta H_\theta=0 \quad (4\text{-}4)$$

$$(\partial_r\lambda)H_r+\frac{1}{r}(\partial_\theta\lambda)H_\theta=0 \quad (4\text{-}5)$$

Let (0, R1, R2, R3) be the inner and outer cylinder respective radii so that $r \in [R_1, R_2]$ and let φ be the relative orientation angle between the cylinders fields as in Fig. 2. We assume that the overall interior field configuration will have the components

$$H_r = H_o(z)\sin(k\theta + \varphi_o(z)) - H_i(z)\sin(k\theta + \varphi_i(z)) \qquad (5\text{-}1)$$

$$H_\theta = -H_o(z)\cos(k\theta + \varphi_o(z)) - H_i(z)\cos(k\theta + \varphi_i(z)) \qquad (5\text{-}2)$$

The exact way the coefficients are connected with the surface magnetizations has been explored in [13].

Condition (7-5) is automatically satisfied for all fields with constant eigen-vorticity λ. Members of this class are often called *Trkalian Flows* in the relevant literature from the name of V. Trkal who first studied them [8]. We further add a constant term which does not affects the first two equations in order to fit (7-3) and (7-4). As we will show next, this corresponds to the addition of constant magnetization terms which are to be computed so as to cancel any z rotation components in the boundaries of the system. Separating the z dependence in the harmonic components the above obtains

$$H_r = A(z)\sin(k\theta) + B(z)\cos(k\theta) \qquad (6\text{-}1)$$

$$H_\theta = C(z)\sin(k\theta) + D(z)\cos(k\theta) \qquad (6\text{-}2)$$

The coefficients are given by

$$\begin{aligned} \{A,D\}(z) &= \pm(H_O\cos\varphi_o \mp H_i\cos\varphi_i) \\ \{B,C\}(z) &= H_O\sin\varphi_o \mp H_i\sin\varphi_i \end{aligned} \qquad (7)$$

From equations (7-1) and (7-2), we get a linear system which leads to

$$\begin{aligned} \dot{C} &= -A, \dot{A} = C \\ \dot{D} &= -B, \dot{B} = D \end{aligned} \qquad (8)$$

The simplest choice results in

$$\begin{aligned} A(z) = B(z) &= h_1^{A,B}\sin(\lambda z) - h_2^{A,B}\cos(\lambda z) \\ C(z) = D(z) &= h_3^{C,D}\sin(\lambda z) + h_4^{C,D}\cos(\lambda z) \end{aligned} \qquad (9)$$

We then derive additional constraints for the cancellation of the third rotation component given by (4-3) and the divergence-free condition (4-4). These two lead to the conditions $C = A, B = -D$ which obtain $h_1^A = h_3^C, h_2^A = -h_4^C, h_1^B = -h_3^D, h_2^B = h_4^D$. Hence the final Beltrami field takes the form

$$H_r = [h_1\sin(\lambda z) - h_2\cos(\lambda z)]\sin(k\theta) + [h_3\sin(\lambda z) + h_4\cos(\lambda z)]\cos(k\theta) \qquad (10\text{-}1)$$

$$H_\theta = [h_1\sin(\lambda z) + h_2\cos(\lambda z)]\sin(k\theta) - [h_3\sin(\lambda z) - h_4\cos(\lambda z)]\cos(k\theta) \qquad (10\text{-}2)$$

Going back to (5) we see that the easiest choice for the relative phase function is to take $\varphi_i = \pm\varphi_o = kz \pm \phi_0$, in which case the constants H_0 must be chosen in accordance with (10). As we only approximate the resulting field with a number of cross-sections corresponding to different

Halbach rings, the choice of the interval in the z axis is arbitrary. We will next try to derive some limits through the analysis of magnetic forces appearing between the two tubes.

3. Beltrami Field Modulators: Spherical Symmetry

In a similar fashion, we may seek for solutions in a spherically symmetric system of closed magnetization shells. Such a system has been known as the *Magic Sphere* [11] and a version of it with very high field density has been constructed recently by Bloch *et al.* [15]. In most cases this is done in order to achieve a very strong homogeneous field (up to 5 Tesla or even more!) in the "z axis" passing through the geometrical north and south poles of the sphere. To this purpose, the direction of each individual dipole moment is made to vary with the polar angle as $\gamma = 2\theta$. This is usually of the form $B_z = (4/3)M \log(\frac{R_{ext.}}{R_{\text{int.}}})$. Instead, we are dealing here with a very different construction of two concentric spheres, having non-uniform magnetizations on their respective shells resulting in a non-homogeneous Beltrami-like field in their interior.

Generic treatment of the interior field of such a sphere leads to a z component of the form

$$B_z = \frac{M}{4\pi r^3}(2\cos\theta\cos\varphi + \sin\theta\sin\varphi) \quad (14)$$

In order to modulate the interior field of two such spheres with inverted fluxes (see Fig. 3) in the form of a Beltrami flow, we need to solve the inverse problem. We thus derive an appropriate expression of the field components which can then be used to find the magnetization on the surfaces of the two spheres. Positioning of the individual dipoles can then be done with numerical methods. To facilitate the derivation we keep a Cartesian coordinate system for the field components and we further assume no z dependence of both B_x and B_y.

$$\partial_y B_z = \lambda B_x \quad (15\text{-}1)$$

$$\partial_x B_z = -\lambda B_y \quad (15\text{-}2)$$

$$\partial_x B_y - \partial_y B_x = \lambda B_z \quad (15\text{-}3)$$

$$\partial_x B_x + \partial_y B_y = 0 \quad (15\text{-}4)$$

$$(\partial_x \lambda) H_x + (\partial_y \lambda) H_y = 0 \quad (15\text{-}5)$$

We restrict our search to Trkalian fields (constant λ) for which (15-5) is automatically satisfied. We also note that taking the crossed derivatives of (15-1) and (15-2) shows that absence of z dependence automatically satisfies (15-4). The three remaining equations are equivalent to the scalar Helmholtz equation

$$(\partial_x^2 + \partial_y^2 + \lambda) B_z = 0 \quad (16)$$

We now prove an important property that reveals the underlying symmetry of (15) and (16). Acting on (15-3) with $\partial_{x,y}$ leads to the diagonalization condition

$$(\nabla_\perp^T \nabla_\perp - \lambda^2 \mathbf{I}) \Omega H_\perp = 0 \quad (17)$$

In the above we have introduced the "transverse" field $H_\perp = [B_x, B_Y]$ and the symplectic Darboux matrices $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \Omega^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ for convenience. Condition (17) shows that the square of the eigen-vorticity is a degenerate eigenvalue of the Hessian for the normal component of the transverse field given that $\mathrm{H}_\perp \Omega \mathrm{H}_\perp = 0$. Therefore a straightforward solution method is to choose $\partial^2_{xy} B_x = \partial^2_{xy} B_y = 0$ and $\partial^2_x B_y = -\lambda^2 B_Y = \lambda \partial_x B_z$, $\partial^2_y B_x = -\lambda^2 B_x = -\lambda \partial_y B_z$. From (15-4) we may also set $\partial^2_x B_x = \partial^2_y B_y = k = const$. The above can be summarized in the two Poisson equations

$$\begin{aligned} \nabla^2 B_x &= (k - \lambda \partial_y) B_z \\ \nabla^2 B_y &= (k + \lambda \partial_x) B_z \end{aligned} \tag{18}$$

These are directly integrated to obtain

$$B_{x,y} = \int \frac{dr'^3}{4\pi^3} \frac{(k \mp \lambda \partial_{y,x}) B_z}{|r - r'|} \tag{19}$$

In order to find the magnetization distribution on the spherical shells it is preferable to analyze (19) in spherical harmonics taking into account the separate boundary conditions for the two shells. Following a similar procedure as the one explained by Halbach in his original papers [8], [9] we may then derive the magnitudes and orientations of the individual dipoles on the each shell surface. We postpone numerical investigation of the above for a separate report.

4. Forces and Torques

The constructions presented in the two previous sections represent rather unusual configurations of surface magnetization which have never been constructed before. As noted in [13], presence of forces and torques is already verified in ordinary concentric Halbach magnetic tubes. While in previous studies, these forces are characterized by certain symmetries, in our case the complexity of the devices requires a separate study. On the other hand, we have to emphasize the fact that the Beltrami flows in general correspond to "force-free" configurations. The term "force-free" comes from the fact that the total Lorentz force $\boldsymbol{J} \times \boldsymbol{B}$ may be cancelled because $\boldsymbol{J} \propto k\boldsymbol{B}$. Even though this may be true for magneto-hydrodynamic flows, in case of permanent magnets will not in general be true due to the fact that an ideal magnetization cannot be realized in a form exactly similar to that of the required field. Caution must be paid to the fact that the magnetizations on the surfaces of the two cylinders or the spherical shells will have to be approximated by many small dipoles.

A straightforward method for evaluation of forces in cases of known magnetic fields is given by direct integration of the electromagnetic tensor through

$$\mathbf{F} = \oint F_{\mu\nu} F^{\mu\nu} ds \tag{20}$$

For the magnetostatic case this is simplified as

$$\mathbf{F} = \mu_0^{-1} \oint [(\mathbf{B} \bullet \mathbf{n})\mathbf{B} - \frac{|\mathbf{B}|^2}{2} \mathbf{n}] ds \tag{21}$$

In what follows we always put $\mu_0 = 1$ as before. We evaluate this integral for the cylindrical Beltrami flux tube. From expression (5-1/2), (11) and (12) we have at least two cases. In each of them, we assume a cylindrical boundary of radius r in the range $R_1 < r < R_2$ enclosing the inner cylinder where integration is to be performed and we compute the force density on a cross section (single pair of rings). Then **n** = [1,0,0] in cylindrical coordinates and

$$\frac{d\mathbf{F}}{dz} = \oint [B_r \mathbf{B} - \frac{1}{2}(B_r{}^2 + B_\theta{}^2)\frac{\mathbf{r}}{|\mathbf{r}|}]ds \tag{22}$$

This is further separated in radial and angular components

$$\begin{aligned} \frac{dF_r}{dz} &= \int_0^{2\pi} \pi r d\theta \left[B_r^2 - \frac{1}{2} B_\theta^2 \right] \\ \frac{dF_\theta}{dz} &= \int_0^{2\pi} 2\pi r d\theta B_r B_\theta \end{aligned} \tag{23}$$

From the total field in (10) we get

$$\begin{aligned} \frac{dF_r}{dz} &= \frac{\pi}{2}\left[A^2(z) + B^2(z) - \frac{1}{2}\left(C^2(z) + D^2(z)\right) \right] \\ \frac{dF_\theta}{dz} &= \frac{\pi}{2}\left[A(z)C(z) - B(z)D(z) \right] \end{aligned} \tag{24}$$

The radial force cancels for a choice of constants given by the constraint $h_1 h_2 - h_3 h_4 = 0$. All forces cancel out when the constants are chosen as $h_1 = h_3, h_2 = h_4$ which represents the true *force-free* case. We also note that the radial force can be made maximal by moving away from the equilibrium point and across the hyperbolic family of curves that correspond to the constraint for the cancellation of the radial force. Thus by increasing the asymmetry of the field we render the system into an unstable region.

5. Beltrami radiation

The simple quasi-static arrangement presented in the previous section, serves to set up an initial state of a Beltrami flow as an equilibrium point of a more general dynamics. We will now describe a mechanism by which it should be possible to modulate the interior flux between the magnetic tubes in order to achieve a controlled variation of the helicity. This type of field modulation is entirely different from existing ones and may be referred to as a type of *Helicity Modulation* (HM).

We can roughly discriminate between four major classes of perturbations of the system described in the previous sections. A) with the addition of a separate pair of Helmholtz Coils enclosing the coaxial Halbach cylinders as in Fig. 2 and applying a purely homogeneous time-dependent component in the z direction, B) by the relative rotational motion of either the interior or the exterior cylinder with respect to the prescribed constant relative angle of π/2 between each cross-section of the cylinders, C) by setting either the outer or the inner tube in linear motion relative to the other and, D) by appropriate modulation of the interior field with a specially designed array of circular currents surrounding the exterior cylinder in such a way that variations of the z component of the internal magnetic flux will result in a kind of longtitudinal variation of the helicity of the flux. The interior magnetic lines should then resemble a number of “spring”-like structures with different contraction factors relative to each other. Given an appropriate control circuit it should be possible to turn the device in a *Helicity Resonator* so that the main wavelength of the longtitudinal variation will coincide with the length of the cylinder.

Any combination of the above is also possible and we believe that it deserves a separate study. It is then possible to examine in controlled situations the type of propagator corresponding to the particular kind of radiation emitted by this instrument in each of the separate cases. One could also add one more type of modulation which is though impossible with permanently magnetized materials. Instead we should use bulk coils in order to be able to change the current directions corresponding to the transformation $k \to -k$. Such alterations from an enclosed flux to an exterior flux are then possible in a way similar to a pulse code modulator.

To further analyze the nature of such an excitation, we refer to the monograph by G. E. Marsh (App. **1**) [2] where it is exactly proven that any magnetostatic solution of the "force-free" equations with constant eigen-vorticity can be used to construct vacuum solutions of Maxwell equations with parallel $\mathbf{E} = \beta\mathbf{B}$ fields. The general case of both spatial and temporal dependency of the eigen-vorticity is not yet completely known and we are not aware of any general results for the exact connection between the two fields.

In general, we cannot guarantee the separability of a vacuum solution in any coordinate system. Instead we may attempt to derive a general criterion for the existence of independent Beltrami-like electric and magnetic components. It is known that the Beltrami problem is equivalent with the solution of a wave or Helmholtz equation [2], so that we may identify its solutions with the vector potential itself. Given that a vector potential **A** exists that satisfies equations (1) and (2), we immediately see that $\mathbf{B} = \lambda(\mathbf{r},t)\mathbf{A}$ and iff $\dot{\mathbf{A}} = \kappa(\mathbf{r},t)\mathbf{A}$ then both the electric and magnetic components will be Beltrami and parallel to $\boldsymbol{A}$ iff also the following conditions hold

$$\nabla\lambda \times \mathbf{A} = \nabla\kappa \times \mathbf{A} = 0 \qquad (27)$$

We may also derive an important constraint between the eigen-vorticities of the electric and magnetic component. For two such parallel fields we should have

$$\begin{aligned} \nabla \times \mathbf{E} &= \lambda_E \mathbf{E} = -\dot{\mathbf{B}} = -\mu\mathbf{B} \\ \nabla \times \mathbf{B} &= \lambda_B \mathbf{B} = \dot{\mathbf{E}} = \kappa\mathbf{E} \end{aligned} \qquad (28)$$

if the conditions $\dot{\mathbf{B}} = \mu(\mathbf{r},t)\mathbf{B}$ and $\dot{\mathbf{E}} = \kappa(\mathbf{r},t)\mathbf{E}$ hold true in a separable system of coordinates. Thus a pair of separate electric and magnetic Beltrami fields will have parallel components $\mathbf{E} = \left(\frac{\lambda_B}{\kappa}\right)\mathbf{B} = -\left(\frac{\lambda_E}{\mu}\right)\mathbf{B}$, with the two eigen-vorticities satisfying

$$\lambda_E \lambda_B + \mu\kappa = 0 \qquad (29)$$

This is a fundamental relation of whom a special case is given by simple monochromatic harmonic fields with $\mu = \kappa = \omega$ as already mentioned in Part I. The above takes as an assumption the separability of the time component in a certain coordinate system. This is not true in general as in the second case where we try to influence directly the eigen-vorticity of a Trkalian field like the one given by (10).

To explain the basic mechanism behind the electromagnetic shutter, we assume a system of two concentric magnetic tubes of which the individual dipoles forming the discretized surface magnetization are formed by appropriately shaped and oriented coils. We are then free to choose the feeding currents so as to conform to either a modulating separable time component or to a certain alteration of the λ parameter of the resulting magnetic field given by (10). In order to find the elements of the resulting electric component we work with the vector potential corresponding to the initial field (10). Indeed, from first principles we have

$$\mathbf{A} = \int dV \frac{\mathbf{B}\times\mathbf{x}}{|\,\mathbf{x}-\mathbf{x}'\,|^3} = \mathbf{z}\int dV \frac{[B_\theta \cos\theta - B_r \sin\theta]}{|\,\mathbf{x}-\mathbf{x}'\,|^3} \qquad (30)$$

We may now easily evaluate the time derivatives of the components in both cases as follows. For a simple harmonic component of the type $[B_r, B_\theta]\exp(\pm\mathbf{i}\omega t)$ we would simply have

$$\mathbf{E} = \pm\omega e^{\pm\mathbf{i}\omega t}\mathbf{z}\int dV \frac{[B_\theta \cos\theta - B_r \sin\theta]}{|\,\mathbf{x}-\mathbf{x}'\,|^3} \qquad (31)$$

In case of a varying eigen-vorticity, we would have to use a special symmetry implied by equations (4-1) and (4-2). We first note that the change of variable $u = \lambda z$ in (10) implies $\partial_t = \dot{\lambda}\partial_\lambda = \dot{\lambda}z\partial_u = (\dot{\lambda}z/\lambda)\partial_z$. This, combined with equations (4-1) and (4-2) gives $\partial_t B_r = \dot{\lambda}zB_\theta, \partial_t B_\theta = -\dot{\lambda}zB_r$ so that the electric component becomes

$$\mathbf{E} = -\dot{\lambda}\mathbf{z}\int dV \frac{z[B_r \cos\theta + B_\theta \sin\theta]}{|\,\mathbf{x}-\mathbf{x}'\,|^3} \qquad (32)$$

In both cases, the vector potential is not a Beltrami flow so that the electric field is not Beltrami-like. There may be more complex types of modulation that also affect the relative angle of the two fields and thus they have a global influence on the way the system radiates in space. In order to further elucidate this subject we provide in the next section, a simple and elegant method for the general construction of such fields based on a separation into transverse and longtitudinal parts.

6. Beltrami fields from symmetries

We extend the previous notions in the more general case by lifting the restriction on the electric and magnetic constants while leaving $c = 1$. We still assume the constitutive relations to be linear. We first show the significance of the sign of these constants and we prove that Beltrami flows can be seen in a general sense as associated with a subgroup of the conformal group. For this we will introduce the symmetric monochromatic fields $\mathbf{F}_\pm = \varepsilon\mathbf{E} \pm \mu^{-1}\mathbf{B}$ for which Maxwell equation in free space obtain

$$\nabla\mathbf{F}_\pm = 0, \nabla\times\mathbf{F}_\pm = \omega\mathbf{F}_\mp \qquad (33)$$

We now observe that any transformation of the electric and magnetic constants of the form

$$\begin{pmatrix}\varepsilon\\ \mu\end{pmatrix} \to \begin{pmatrix}\varepsilon'\\ \mu'\end{pmatrix} = \begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\begin{pmatrix}\varepsilon\\ \mu\end{pmatrix} \qquad (34)$$

may turn (33) into a Beltrami flow $\nabla\times\mathbf{F}_\pm = l\omega\mathbf{F}_\pm$ if the condition $(\varepsilon', \mu') = l(\varepsilon', -\mu')$ holds true in which case $\lambda_\pm = \{\omega, l\omega\}$. This implies a constraint on the matrix elements which results into dyadic matrices of the form

$$\mathbf{G} = \begin{bmatrix}\alpha & \beta\\ l\alpha & -l\beta\end{bmatrix} = \begin{pmatrix}1\\ l\end{pmatrix}\begin{pmatrix}\alpha & \beta\end{pmatrix} \qquad (35)$$

We immediately recognize that the same transformation could have been applied at the field “vectors” $[\mathbf{E}, \quad \pm\mathbf{B}]$. Hence, there should exist a mapping

$$\mathbf{F}\pm \rightarrow \mathbf{F}'_{\pm} = \mathbf{GF}\pm : \mathbf{F}_{+} = l\mathbf{F}_{-}, \nabla \times \mathbf{F}_{\pm} = \omega \mathbf{F}_{\pm}. \qquad (36)$$

We now think of the general case of two parallel Beltrami flows representing the electric and magnetic components of a triplet {***E, B, J***} of solutions of Maxwell equations compatible with (1) and (2), with all three components non zero. We may separate them into transverse and longtitudinal parts as

$$\mathbf{E} = \mathbf{E}_{\perp} + \mathbf{E}_{Z}$$
$$\mathbf{B} = \mathbf{B}_{\perp} + \mathbf{B}_{Z}$$
$$\mathbf{J} = \mathbf{J}_{1} + \mathbf{J}_{2}$$

Due to the linearity of Maxwell equations we may now recognize two triplets $\{\mathbf{E}_{\perp}, \mathbf{B}_{Z}, \mathbf{J}_{1}\}$ and of $\{\mathbf{E}_{z}, \mathbf{B}_{\perp}, \mathbf{J}_{2}\}$ of independent solutions which should in general correspond to two different physical systems. We also know that Maxwell equations are invariant to the duality transformation $\mathbf{E} \rightarrow \mathbf{B}, \mathbf{B} \rightarrow -\mathbf{E}$ either in free space or with the inclusion of magnetic sources. Assume now that the first triplet contains normal solutions of Maxwell equations (not Beltrami) for which $\mathbf{E}_{\perp} \bullet \mathbf{B}_{z} = 0$. Then the same must hold true for the second triplet as well in which case the simplest method to produce a pair of parallel Beltrami-like electric and magnetic components is to take an arbitrary triplet of solutions $\{\mathbf{E}_{\perp}, \mathbf{B}_{Z}, \mathbf{J}_{1}\}$, apply a duality transformation to get the second triplet $\{-\mathbf{B}_{Z}, \mathbf{E}_{\perp}, \mathbf{J}_{2}\}$ where either $\mathbf{J}_{2} = 0$ or

$$\mathbf{J}_{2} = -(\nabla \times \mathbf{E}_{\perp} + \mathbf{B}_{Z}) = \nabla \times \mathbf{M} \qquad (37)$$

from which we may compute the necessary magnetization on the boundaries of the system through the inverse Biot-Savart operator.

We can combine these symmetries using a simple example in free space of a peculiar vacuum solution first found by Chubykalo [16]. Let the electric and magnetic components be given as

$$\mathbf{E} = \theta \left[\frac{\omega a \sin\theta}{r^{2}} \right] \sin(\omega t - \varphi_{0})$$
$$\mathbf{B} = \left[\mathbf{r} \left(\frac{2\alpha \cos\theta}{r^{3}} \right) + \theta \left(\frac{(\beta - \alpha)\sin\theta}{r^{3}} \right) \right] \cos(\omega t - \varphi_{0})$$
$$\alpha = \cos(\omega r - \delta) + \omega r \sin(\omega r - \delta)$$
$$\beta = \omega^{2} r^{2} \cos(\omega r - \delta)$$
$$\delta = (\nu + \frac{1}{2})\pi$$

This solution exhibits a kind of "self-quantization" with successive spherically symmetric energy shells filling all space. Applying the duality transformation explained before we end up with the new set of fields $\{\mathbf{E}' = \mathbf{E} - \mathbf{B}, \mathbf{B}' = \mathbf{E} + \mathbf{B}\}$ which are no more vacuum fields but requires the presence of magnetization sources satisfying $\nabla \times (\mathbf{E}' + \mathbf{M}) = \mathbf{B}'$. Applying now the transformation (35) we get

$$\begin{pmatrix} \mathbf{E}'' \\ \mathbf{B}'' \end{pmatrix} = \begin{bmatrix} \alpha + \beta & -(\alpha - \beta) \\ l(\alpha - \beta) & l(\alpha + \beta) \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \mathrm{E} \\ \mathrm{B} \end{pmatrix} \qquad (38)$$

We immediately recognize that the new transformation matrix can be analyzed in a scaling factor given by $\Lambda = l(\alpha^2 + \beta^2)$ and a Moebius transformation with Det = 1. The new fields should correspond to Beltrami flows as in (36) with the sources also transformed as

$$\begin{pmatrix} 0 \\ \mathbf{J} \end{pmatrix} \rightarrow (\alpha + \beta) \begin{pmatrix} \mathbf{J} \\ l\mathbf{J} \end{pmatrix} \qquad (39)$$

We see that the total transformation demands also the addition of a time-varying magnetization term of the order $\partial_t \mathbf{M} \propto (\alpha + \beta)\mathbf{J} \propto \nabla \times \mathbf{M}$. Thus we see that a wise choice is to take the magnetization to be a Beltrami field itself. On the other hand, **E''** = β**B''** together with the condition $\nabla \times (\mathbf{E}'' + \mathbf{M}) = \mathbf{B}''$ lead to $\nabla \times \mathbf{M} = (1 - \lambda\beta)\mathbf{B}''$ from which the simplest choice is to take $\mathbf{M} = (\lambda^{-1} - \beta)\mathbf{B}''$.

It deserves to mention that the original solution of Chubykalo admits decomposition in two counter-propagating wave-like solutions. In our case, the transformed solution is necessarily a kind of 4-wave interaction. Such a phenomenon is associated with the very interesting area of "*phase conjugation*" [17] which results into an apparent time-inversion ("*retro-reflection*" in real-time holography) of the propagating waves thus canceling their dispersion and resulting in energy focusing. This requires the presence of non-linear interactions resulting in frequency mixing inside a material with appropriate polarization tensor. Therefore, the possibility of building the appropriate boundary conditions for a spherical chamber causing such a phenomenon could have important implications in plasma and fusion research. We postpone further examination of these matters for a later publication.

We now turn back at our initial observation that the combined transformation of the fields is isomorphic with a transformation of the electric and magnetic components such that if $\mathbf{z} = [\varepsilon, \mu^{-1}]$ then $\mathbf{z} = \pm\Lambda\mathbf{G}\mathbf{z}$ where **G** is a member of the Moebius group

$$\mathbf{G} = \Lambda^{-1} \begin{bmatrix} \alpha - \beta & \alpha + \beta \\ -l(\alpha + \beta) & l(\alpha - \beta) \end{bmatrix} \qquad (40)$$
$$\Lambda = l(\alpha^2 + \beta^2), \alpha \neq \pm\mathbf{i}\beta$$

The new fields are then given as $\mathbf{F}_\pm = \varepsilon_\pm \mathbf{E} + \mu_\pm^{-1}\mathbf{B}$. This particular form is conjugated to the so called *elliptic* transform when 0 < *l* < 4 and $\alpha + \beta = \sin\gamma, \alpha - \beta = \cos\gamma$ with γ an arbitrary parameter. We can generalize this result by introducing the Beltrami-Maxwell-Lakhtakia complex fields $\mathbf{Q}'_\pm = \varepsilon\mathbf{E} \pm \mathbf{i}\mu^{-1}\mathbf{B} \xrightarrow{\mathbf{G}} \mathbf{Q}'_\pm = \varepsilon_\pm\mathbf{E} + \mathbf{i}\mu_\pm^{-1}\mathbf{B}$ in which case we are allowed to make full use of the complex Moebius transform. This more general category corresponds to arbitrary materials with a complex index of refraction a theme which has been extensively studied by Lakhtakia [4] and others.

On the other hand, the method we prescribed for building arbitrary Beltrami flows from an initial set of solutions of Maxwell equations implies that the distinction between ordinary normal electric and magnetic components and the parallelized Beltrami fields is essentially based on our interpretation of the properties of the vacuum. In other words, it seems meaningless to consider one of the two categories as more fundamental than the other.

It is though much more interesting to connect this with the so-called *Polarizable Vacuum Rerpresentation* of general relativity which is based on a direct application of the Lorentz transform into the electric and magnetic constants. This approach has been initially proposed by Wilson in an article of 1922 to be rediscovered by Dicke at 1957 [20] and it was recently reviewed by Putthoff [21] in connection with his electromagnetic mass theory. We recall that the Lorentz group is contained as a subgroup in the more general Moebious geometry which corresponds to the compactification of a Lorentzian manifold. This also resonates with the first Part of this report where we emphasize the importance of the point *Ad Infinitum* for a geometric interpretation of the origin of Maxwell equations.

7. Helicity Modulation

In what follows, we choose to define as "Helicity Modulation" the temporal or spatial variation of the topology of electric and magnetic components of a Beltrami flow. It is not strictly restricted in such a case of fields and there may be other types of solutions that might also be used. We can propose two possible mechanisms for achieving such an effect and we will use the magnetic tubes with cylindrical symmetry to explain the basic mechanism.

In the first case, we just add a time component by a direct influence on the source currents. In the second case, we affect the eigen-vorticity λ either through an external signal or by means of combined mechanical motions. The simplest means for inducing such an alteration given certain symmetry conditions is shown in Fig. 4 where, the inner and the outer tubes are at distances $\left[-z_0/2,+z_0/2\right]$ apart from a common centre and they enter one inside the other with a velocity *v.*

We may resemble this mechanism with an *"Electromagnetic Shutter"* composed either of mechanically movable tubes or by immovable parts with an electronic controller which sends an electric current front in a set of coils on the surfaces of the two tubes instead of moving them. There are certain engineering advantages and disadvantages in choosing between these two constructions, the most important being the amount of energy transferred in the propagating disturbance. This in turn depends crucially on the excitation method used.

We recall here that the total helicity of the electromagnetic field is a conserved quantity given by

$$\mathbf{h} = \int dV \mathbf{A}\mathbf{B} = \int dV \oint dV' \mathbf{B} \bullet [\mathbf{B} \times \nabla \| \mathbf{r} - \mathbf{r}' \|] \qquad (41)$$

The differential expression for the helicity density takes the form of a continuity equation

$$\frac{\partial \mathbf{h}}{\partial t} = -\nabla(\mathbf{E} \times \mathbf{B} + \chi \mathbf{B}) - 2\eta \mathbf{J}\mathbf{B} \qquad (42)$$

Apparently, any variation of the total helicity in the interior of the instruments described in sections 2 and 4, should be accompanied by an equal and opposite reaction of the exterior field.

We restrict our attention to the helicity of Beltrami fields. Using the main results of section **5** we see that for true Beltrami electric and magnetic components the magnetic field becomes the sole contributor to the helicity density. We also note that

$$\nabla \bullet (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \bullet (\nabla \times \mathbf{E}) - \mathbf{E} \bullet (\nabla \times \mathbf{B}) = (\lambda_E - \lambda_B) |\, \mathbf{E} \,\|\, \mathbf{B} \,| \qquad (43)$$

The fact that there is still a divergence even for parallel electric and magnetic components is a general result for Beltrami flows not restricted to electromagnetism. Using again the results of section **5,** we see that in free space the helicity density of two parallel fields is expected to vary like

$$\mathbf{h} = -< (\mu + \kappa^{-1} \lambda_B^2) |\, \mathbf{B} \,|^2 >_T \qquad (44)$$

For simple harmonic Trkalian fields we get $(\omega^2 + \lambda^2)/\omega = \lambda_H = const.$ For the simple case of the magnetic tubes presented in sections **2** and **3,** only the magnetic field represents a Beltrami flow so that

$$\nabla \bullet (\mathbf{E} \times \mathbf{B}) = -\mathbf{B} \bullet \dot{\mathbf{B}}, \mathbf{E} \bullet \mathbf{B} = 0$$

For a simple harmonic modulation of the magnetic term we should have

$$\mathbf{h} = -\omega <|\mathbf{B}|^2>_T \quad (45)$$

We note also that the above can be rewritten with the aid of purely statistical measures for the field time distribution at a point in space allowing for a modulation – demodulation scheme to be developed. By introducing the dispersion $\sigma =<\mathbf{B}>^2 - <|\mathbf{B}|^2>, \mu =<\mathbf{B}>$ the above relation becomes

$$\mathbf{h} = \omega(\sigma - \mu^2) \quad (46)$$

The above have a simple practical implication. For normal electric and magnetic components, when the signal is buried under a noisy background ($\sigma >> \mu$), it is only possible to recover it by adjusting the field strength in a region where $\sigma - \mu^2 < 0$. Another way would be to change the sign of helicity through the use of very strong pulses. We assume here that the detection instruments should be sensitive to polarization so that an appropriately shaped phased array could be adequate. On the other hand, parallel electric and magnetic components offer the advantage that a variation of the eigen-vorticity (topological characteristics) may still render a very weak signal perceptible.

At this point, a number of observations are adequate. Although the helicity currents can be properly defined through the product **EB**, the conserved helicity integral usually comes through the product **AB**. This last term has often resulted in confusion and it seems that this is also inherently connected with topological problems in multiply connected domains nad the important distinction between *twisting* and *linking* helicity. In [2] G. E. Marsh (p. 69-70) presents an example of a magnetic field with explicit twisting which the vector potential definition fails to grasp! On the other hand, it is probably impossible to artificially create some of these configurations as they present the problem of infinite current sources. A plausible answer would be to allow the use of plasma currents in order to approximate them locally inside a region.

Last but not least, a very important question should be posed concerning the possibility of artificially creating Beltrami fields with arbitrary variations not only of the helicity but also of the angle between the electric and magnetic components. We first recognize that for every two real fields **E** and **B** in R^3 we should be able to write in general

$$|\mathbf{E} \bullet \mathbf{B}|^2 + |\mathbf{E} \times \mathbf{B}|^2 = |\mathbf{E}|^2 |\mathbf{B}|^2 \quad (47)$$

What we mean by angle modulation is the possibility of a continuous alteration between the two extreme cases of normal ($|\mathbf{E} \bullet \mathbf{B}|^2 = |\mathbf{E}|^2 |\mathbf{B}|^2$) and parallel ($|\mathbf{E} \times \mathbf{B}|^2 = |\mathbf{E}|^2 |\mathbf{B}|^2$) electric and magnetic components. It is known already that such a case is in fact isomorphic with the continuous application of a transformation from the Poincare group. Both, Benn and Kress [18] as well as Pantilie and Wood [19], agree that the case of non-constant eigen-vorticity corresponds to certain curved metrics and in the second article it is also shown the equivalence with a certain class of conformal metrics of the form $g = rh + r(dr + A)^2$. In such a case the vector potential itself may be a solution of the Beltrami equations (1) and (2) but the non-separable nature of the electric and magnetic components would fail to satisfy the additional conditions (28) and (29) so that the polarization angle may become arbitrary. In simple terms, any device that could modulate the eigen-vorticity so as to control the angle of the electric and magnetic components would be equivalent to an *Electromagnetic Warp Engine*.

Using the two fundamental invariants of the Electromagnetic field we may rewrite the rhs of (47) in terms of the invariants of the field through

$$|\mathbf{E}|^2 |\mathbf{B}|^2 = -\frac{1}{2}[(|\mathbf{E}| - j|\mathbf{B}|)^2 - u_{EM}]^2 = -\frac{1}{2}[(|\mathbf{E}| + j|\mathbf{B}|)^2 - P]^2 \quad (48)$$

where $Q = \mathbf{E} \bullet \mathbf{B}$ and $P = |\mathbf{E}|^2 - |\mathbf{B}|^2 = (|E| + j|B|)(|E| - j|B|)$ the electromagnetic invariants. By introducing two scalars $\Phi_{\pm} : P = -\Phi_{+}\Phi_{-}$, we may rewrite the identity (47) in terms of P and Q as

$$Q^2 + |\mathbf{S}|^2 = |(\Phi_+^2 - u_{EM})\frac{e^{j\omega}}{2}|^2 = |(\Phi_-^2 - P)\frac{e^{j\omega}}{2}|^2 \qquad (49)$$

It is customary to use the index $I(P,Q) = P^2 + Q^2$ for classification of EM fields []. Null fields always satisfy $I = 0$. It is generally known that Beltrami fields with non-constant eigen-vorticity contain non-null fields for which $Q \neq 0$. The simplest case in a flat space is given by two pairs of plane conjugate waves with Q > 0 [2].

Assuming that a helical flux tube of the Beltrami type can be formed in the interior of the magnetic cylinders of the previous section, it is possible that the orientation and the relative angle of the electric and magnetic components may vary locally with space and time coordinates. Under extreme conditions this could also be associated with the so called, Rainich [22] and Melvin [26],[27] *electrovac* solutions. In such a situation the geometric validity of relation (47), which is just a trigonometric relation of the form $\cos^2(E\square B) + \sin^2(E\square B) = 1$, could be cast in doubt at least locally. It is therefore possible to provide a number of experimental tests of such conditions for the class of devices presented here under extreme conditions of very strong currents and strong magnetic fields. It is probable that for such experiments ordinary magnetization sources should be replaced with superconducting coils. The subject of angle/eigen-vorticity modulation will have to be investigated further in a separate publication.

6. Conclusions

In this report we have investigated several possible methods for building instruments capable of approximating force-free solutions of Maxwell equations. For this we started from a desired vacuum solution and tried to locate the simplest types of fields with cylindrical and spherical symmetry that could be used to locate the appropriate boundary conditions via an inverse method. Numerical treatment will have to be based on the appropriate expansions of the Biot-Savart operator in these two systems through which an approximation for the surface magnetizations and charge sources of the system boundaries could be obtained similar to a procedure first proposed by K. Halbach.

Furthermore, we have proposed a variety of possible methods for exciting time-dependent solutions through which a modulation of the helicity content and the field eigen-vorticity could be controlled and modified according to an external signal. In such a case it is probable that the use of bulk coils might be preferable instead of permanent magnets at the cost of increasing complexity. These methods can be extended with extensive use of special symmetries of vacuum Maxwell equations in order to increase the range of possible applications of this technique into an arbitrary set of ordinary solutions via a transformation which is inherently connected with the conformal group.

It is our conclusion that this connection is *deep* and requires further investigation as it seems to be connected with many other fundamental themes, part of which has been explained in the first part of this report, and especially with certain kinds of curved metrics.

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Figures

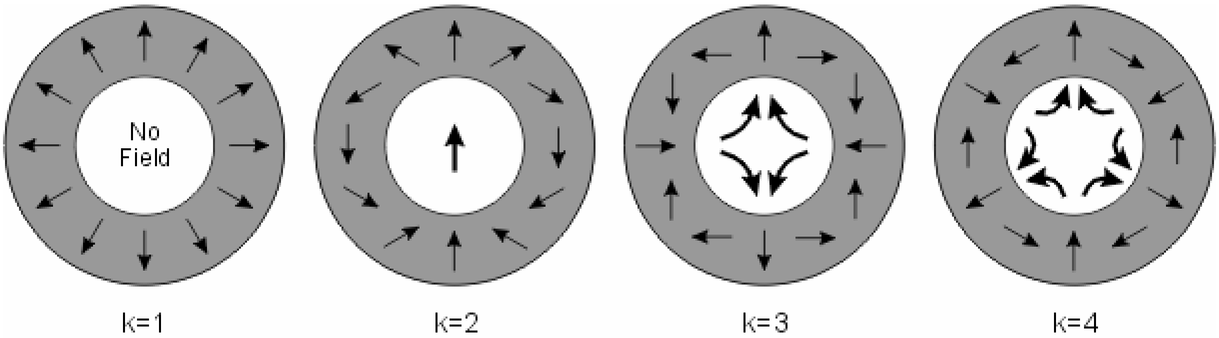

Fig. 1

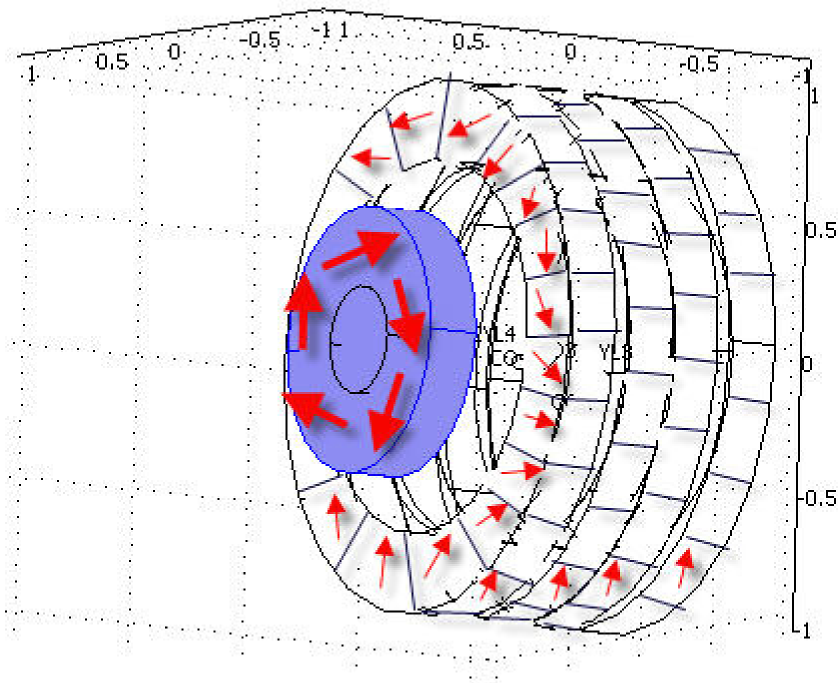

Fig. 2

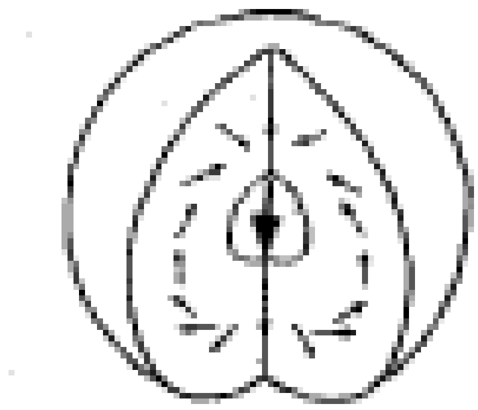

Fig. 3

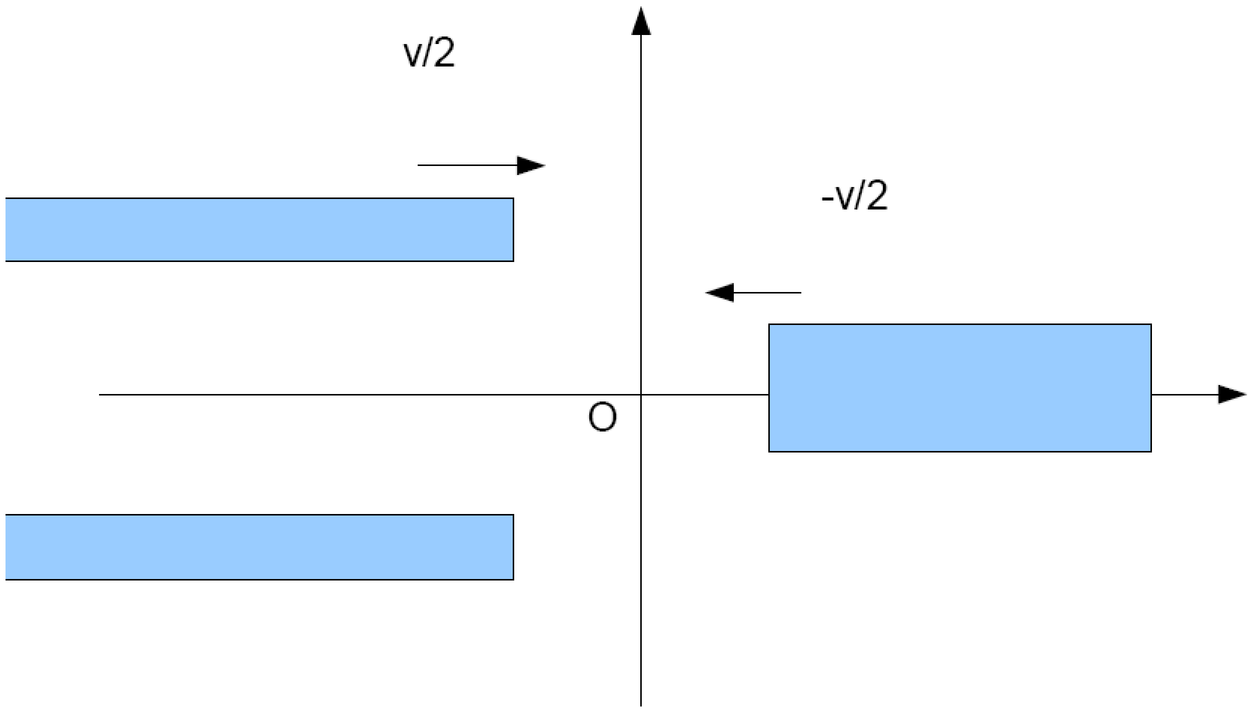

Fig. 4