The Generalized Curvature of Locally Convex Type on Manifolds M^d in R^n

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0. Introduction

In previous papers [2], [3], P. Stavrinos defined certain different concepts which he called curvatures of locally convex type $K^{(1)}_{lct}$, $\Lambda^{(1)}_{lct}$. The two most fundamental ones are defined on surfaces immersed in IR^3.

In this paper the definitions of the curvatures of locally convex type are made more general. Their application is also extended to hypersurfaces in IR^n. Therefore we study not only surfaces in R^3 but also curves in IR^2 or IR^3 etc.

Also we define the concept and study the properties of $\Pi(\Sigma)$, the set of all points p ε IR^n such that the curvature of locally convex type of a hypersurface Σ of dimension d=1,…(n-1) is different from zero in p.

1. Basic Definitions

***1.1.* Definition** *Let M^d an immersed manifold in IR^n with d=1,2…n-1, we define the $K^{(1)}_{l.c.t.}$ – curvature of locally convex type*

$K^{(1)}_{lct}(p, M^d) = 1/R^{(1)}$

where $R^{(1)} = \inf\{r > 0 \,/\, \overline{B}^n(p, r) \cap M^d$ not connected, where $p \in IR^n\}$

Here $\overline{B}^n(p, r)$ signifies a closed ball in IR^n.

***1.2.* Definition** *Let M^d as above, we define the $\Lambda^{(1)}$ – curvature of locally convex type at a point $p \in IR^n$ the quantity $\Lambda^{(1)}_{lct}$ with $R^{(1)}=\inf\{r > 0 \,/\, S^{n-1}(p, r) \cap M^d$ not connected $\}$*

1.3. Definition *Let M^d as above and let $p \in M^d$. We will say that M^d is of <u>locally convex type I</u>, if $\forall p \in M^d \, \exists r_0 > 0$ such that if $0 < r < r_0$, then $\overline{B}^n(p, r) \cap M^d$ is the closure of a simply connected neighborhood of all its interior points.*

From now on, all our M^d 's will be of locally convex type I, unless explicitly stated otherwise. The most important reason for the extended definitions, is not so much that the dimension of M^d and IR^n are now more general than two or three but that the point p does not necessarily belong to M^d but is an arbitrary point of IR^n.

By definition 1.1. the points of R^n are divided into two sets with respect to M^d.

a) The set of points p for which $K^{(1)}_{l.c.t}(p, M^d) = 0$
b) The set of points p for which $K^{(1)}_{l.c.t} > 0$.
   This last set is the most important for the understanding and use of the concept of $K^{(1)}_{l.c.t.}$.

1.4 Definition *We call $\Pi(M^d)$ the set of all points $p \in IR^n$ for which $K^{(1)}_{l.c.t}(p, M^d) > 0$.*

2. The properties of $\Pi(M^d)$

We now give some theorems on the above definitions.

2.1. **Theorem** *A necessary and sufficient condition for a point $p \in IR^n$ to belong to $\Pi(M^d)$ is the following:*

a) There exists a closed subset $Q \subset M^d$ which disconnects M^d, that is $M^d \setminus Q$ is not connected.

b) There exist two points $q_1, q_2 \in M^d$ which belong to different connected components of $M^d \setminus Q$ and $\forall q \in Q$ $\max\{|pq_1|, |pq_2|\} < |pq|$.

Proof: i) If statements (a) and (b) are valid then it follows that $K^{(1)}_{l.c.t}(p, M^d) \neq 0$. To prove that, it is easy to see that a ball $\overline{B}^n(p,r)$ with radius r, such that $\forall q \in Q$ $\max\{|pq_1|,|pq_2|\} \leq r < |pq|$ will have a not connected intersection with M^d, since $\overline{B}^n(p,r) \cap M^d$ will contain q_1, q_2 but not any points $q \in Q$. So the set $\{r>0 |\ \overline{B}^n(p,r) \cap M^d$ not connected$\}$ is not empty. Therefore, it has an infimum less than infinity and the inverse of this infimum is not zero, that is $K^{(1)}_{l.c.t}(p, M^d) \neq 0$.

ii) Inversely, let $K^{(1)}_{l.c.t}(p, M^d) \neq 0$, then a number r exists such that if $T = \overline{B}^n(p,r) \cap M^d$, then T is not connected and let $q_1, q_2 \in T$ but let them belong to different connected components of T. Let $q_1 \in Q_1$, where Q_1 is the connected component of q_1, and $q_2 \in Q_2$, where $Q_2 = T \setminus Q_1$. Since T is not connected, we find two open sets $U_1, U_2 \subset M^d$ such that $U_1 \cap U_2 = \emptyset$, $q_1 \in Q_1 \subset U_1$ and $q_2 \in Q_2 \subset U_2$. Because $U = U_1 \cup U_2$ is open then $Q = M^d \setminus U$ is closed so, Q disconnects M^d, since $M^d \setminus Q = U_1 \cup U_2$. Also since $\overline{B}^n(p,r) \cap Q = \emptyset$ we have $\forall q \in Q$ $\max\{|pq_1|,|pq_2|\} \leq r < |pq|$.

We now give another theorem which describes an important property of $\Pi(M^d)$.

2.2. **Theorem** *$\Pi(M^d)$ is an open subset of IR^n*

In order to prove theorem 2.2. we first need a simple Lemma

2.3. Lemma *Let $p \in IR^n$ and M^d as above. Let $q, q' \in M^d$. Also let $|pq'| < |pq|$ then there is an open ball $\overset{o}{B}{}^n(p,\varepsilon)$ such that $\forall p' \in \overset{o}{B}{}^n(p,\varepsilon)$, $|p'q'| < |p'q|$.*

Proof: Let $|pq| - |pq'| = \delta$ and $|pp'| < \varepsilon$. Then $|pq| - \varepsilon < |pq| - |pp'| \leq |p'q|$ and $|p'q'| \leq |pq'| + |pp'| < |pq'| + \varepsilon$. Now, if $|pq'| + \varepsilon < |pq| - \varepsilon$, we will have the desired result.

For this, it is sufficient to choose $2\varepsilon < \delta = |pq| - |pq'|$ or $\varepsilon < \delta/2 = (|pq| - |pq'|)/2$.

So, if $\varepsilon < \delta/2$, then $\forall p' \in \overset{o}{B}{}^n(p,\varepsilon)$, $|p'q'| < |p'q|$.

Now we can prove theorem 2.2.

Proof of Th. 2.2. By Lemma 2.3 we can see that if statements (a) and (b) of theorem (2.1) are valid for a point $p \in IR^n$ then there is an open ball $\overset{o}{B^n}(p,\varepsilon)$, such that, $\forall p' \in \overset{o}{B^n}(p,\varepsilon)$ we have

a) Q disconnects M^d

b) $\forall q \in Q$, $\max\{|p'q_1|,|p'q_2|\} < |p'q|$

Therefore $\overset{o}{B^n}(p,\varepsilon) \subset \Pi(M^d)$, $\forall p \in \Pi(M^d)$, so $\Pi(M^d)$ is open in IR^n.

Now we can give another theorem for a necessary and sufficient condition for $p \in IR^n$ to belong to $\Pi(M^d)$, based on the following definition.

***2.4.* Definition** *Let Q be a closed subset of M^d. We will say that Q is a set of* *maximum distance* *from p if and only if there is an open neighborhood N(Q) such that $\forall q \in Q$ and $\forall q' \in N(Q) \backslash Q$, $|pq'| < |pq|$.*

***2.5.* Theorem** *A necessary and sufficient condition for a point $p \in IR^n$ to belong to $\Pi(M^d)$ is that there exists a closed subset $Q \subset M^d$ which:*

a) Disconnects M^d

b) Is a set of maximum distance from p.

Proof: i) Let conditions a) and b) hold: Then, since Q disconnects M^d it will also disconnect N(Q). To prove this we observe that since Q is closed, $T = M^d \backslash Q$ is open. T is not connected, so it will contain two non empty, disjoint, open subsets U_1, U_2 and $U_1 \cup U_2 = T$. Now $N(Q)\backslash Q = N(Q) \cap T = N(Q) \cap (U_1 \cup U_2) = (N(Q) \cap U_1) \cup (N(Q) \cap U_2) = N_1 \cup N_2$. So N(Q) contains points of both U_1 and U_2 and N_1, N_2 are not empty, since the distances of U_1 and U_2 from Q are zero. Then we can say we have two points $q_1 \in N_1$, $q_2 \in N_2$ which belong to different connected components of T and for which we have:

$\text{Max}\{|pq_1|,|pq_2|\} < |pq| \ \forall q \in Q$.

Therefore by theorem 2.1. $p \in \Pi(M^d)$.

ii) Let $p \in IR^n$ and M^d as above and let $K^{(1)}_{l.c.t}(p, M^d) > 0$. Then $\exists r > 0$ such that $\overline{B}^n(p,r) \cap M^d$ not connected. We can prove that there is an $r' > 0$, $r < r'$ such that $\overline{B}^n(p,r') \cap M^d$ is connected. Otherwise we could define two open subsets $U_{1,r}$, $U_{2,r}$ of M^d so that $U_{1,r} \cap U_{2,r} = \emptyset$ and each of them would contain a part of $\overline{B}^n(p,r') \cap M^d$. Then $\forall r' \in (r, +\infty)$ we could define $A_1 = \cup U_{1,r'}$, $A_2 = \cup U_{1,r'}$ and obviously: A_1, A_2 open subsets of M^d, $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = M^d$.

That means M^d would be not connected. This is obviously not true by definition of M^d. Therefore: $\exists\ r' > 0$, $r < r'$, such that $\bar{B}^n(p,r') \cap M^d$ connected.

Now, there must be an $r_1 > 0$ such that $r_0 < r < r_1$, means that:

$\bar{B}^n(p,r) \cap M^d$ non connected where r_0 is given by definition 1.3 and there must bed an $r_2 > 0 : \forall r' > 0$, $r_1 < r' < r_2$, $\bar{B}^n(p, r') \cap M^d$ connected.

Then we define a set $Q \subset M^d$ as follows:

$Q = \{q \in M^d / r_1 \leq |pq|\}$.

Q is a closed subset of M^d, because it's complement:
$A = \{q' \in M^d / |pq'| < r_1\}$
is open.

To prove this, we observe that $\forall q' \in A$ and $\forall q'' \in \overset{o}{B}{}^n(q',\varepsilon) \cap M^d$ where:
$|pq'| + \varepsilon < r_1$
we have:
$|pq''| \le |pq'| + |q'q''| < |pq'| + \varepsilon < r_1$.

Therefore $q'' \in A$ and $\overset{o}{B}{}^n(q',\varepsilon) \subset A$.
Now we can call $M^d = N(Q)$ and we observe that Q

a) is a set of maximum distance from p.
b) disconnects M^d.

Property (b) can be proved if we consider again sets $U_{1,r}$, $U_{2,r}$ as above, where $r_0<r<r_1$. Then $A_1' = \cup U_{1,r}$, $r \in (r_0, r_1)$, $A_2' = \cup U_{2,r}$, $r \in (r_0, r_1)$, are open subsets of M^d and we have:
$A_1' \cap A_2' = \emptyset$, $A_1' \cup A_2' = M^d \setminus Q$.
So, $A = M^d \setminus Q$ is non connected. That concludes the proof of theorem 2.5.

3. The study of Generalized Curvature $K^{(i)}_{l.c.t}$ in M^1

In this work so far, we have considered manifolds M^d immersed in IR^n, as defined in Spivak, ([1] Vol. I, p.p. 1-15, 2-26, 2-27) which are locally homeomorphic to IR^d or H^d ([1] Vol. I, p.p. 1-22, 1-23)
They are supposed to be differentiable their order of differentiation being at least C^1 or higher, if necessary and if so specified, and finally being M^d of locally convex type I as in definition 1.3. All results stated so far are valid for any such manifolds, without special distinctions.
Now, we would like to present some theorems, concerning one dimensional manifolds, M^1, i.e. curves, immersed in IR^n. In order to state and prove these theorems, it will be necessary, among other things to distinguish between curves with, or without a boundary.
Since we assume that our curves are of locally convex type-I they do not contain double points. Therefore ([1], Vol. I p.p. 1-8) there are two classes of connected curves:

a) Homeomorphic to connected one dimensional subsets I of IR. Such as IR, IR^+ or [0,1].
b) Homeomorphic to S^1.

We can immediately state a simple corollary of our previous results

***3.1.* Corollary** *Let $M^1 \subset IR^n$ be a curve as in (a) above, then a necessary and sufficient condition for a point $p \in IR^n$ to belong to $\Pi(M^1)$ is the following: there are three successive points $q_1, q_3, q_2 \in M^1$ such that: $\max\{|pq_1|, |pq_2|\} < |pq_3|$ with $M^1(t_1) = q_1$, $M^1(t_2) = q_2$, $M^1(t_3) = q_3$ and $t_1 < t_3 < t_2$.*

<u>Proof:</u> This is simply an application of theorem 2.1 where the points q_1, q_2 have exactly the same meaning, and $Q=\{q_3\}$, as in the same theorem.

The next corollary is equally obvious. In order to formally state it, though, we need some more observations and new definitions. First of all the distance of the point q_3 from p is larger than that of q_1 or q_2. Therefore it is only natural that we seek the "best" q_3 available. This should be a q_3 with a maximum

distance $|pq_3|$ compared to other points in it's neighborhood in the sense of definition (2.4), where $Q=\{q_3\}$. But, such an isolated point is not always available. It might happen that p is the centre of a circle of radius r, that a connected, closed arc Q of this circle is part of M^1 and Q is a set of maximum distance from p in the sense of definition (2.4). So we must declare that Q, the set of maximum distance, is either a point q_3 or an arc of a circle Q with p as a centre.
Second, the points q_1, q_2 can be defined to be of minimum distance from p. Since they don't have to disconnect M^1, they can be boundary points of M^1, if M^1 has such points. Also if M^1 does not contain its boundary points, then it may be that $q_1, q_2 \in R^n$ do not belong to M^1 at all, but $d(q_1, M^1)=d(q_2, M^1)=0$ where d is the distance of a point from M^1 as a subset of R^n according to the metric of R^n. In that case a relation such as $t_1 < t_2 < t_3$ has no meaning since $q_1, q_2 \notin M^1$, so they are not the images of some $t_1, t_2 \in R$ through a function M^1: IR $\rightarrow$ IR^2.
To take all these considerations into account, we must give some new definitions concerning manifolds $M^d \subset IR^n$ in general and curves $M^1 \subset IR^n$ in particular.

3.2. Definition We will call a set $Q \subset M^d$ <u>a set of constant maximum distance</u> of $p \in IR^n$ from M^d, if:

(1), a) Q is a set of maximum distance from p

b) $\forall p \in Q$, $|pq|$ is constant.

(2) We will call $Q \subset M^d$ a set of constant minimum distance from p if

a) Q is a set of minimum distance, that is $\exists$ N(Q) an open neighborhood of Q such that $\forall q \in Q$ and $\forall q' \in N(Q) \setminus Q$, $|pq| < |pq'|$.

b) $\forall q \in Q$, $|pq|$ is constant.

Obviously in either case Q contains either a single point q or a proper subset of $S^{n-1}(q,|pq|)$ where $q \in Q$ which is closed in M^d.

(3) Let $M^d \subset R^n$ as above and $Q \subset IR^n$. We will call Q a set of constant supremum distance from $p \in IR^n$ if

a) $\forall q \in Q$ $d(q, Md) = 0$, so either $q \in M^d$ or q is a boundary point of M^d which does not belong to M^d.

b) $\forall q \in Q$, $\exists r_q > 0$ where $0 = d(q, M^d) < r_q < r_{oq}$

Where r_{oq} as in def (1,30 and $[(\cup_{q\in Q} \bar{B}^n(q, r_q)) \cap M^d]$ is the closure of an open simply connected neighborhood of all its points $q' \in Q \cap M^d$.

c) $\forall q \in Q$, $|pq|$ is constant

d) if $N_Q = ((\cup_{q\in Q} \bar{B}^n(q, r_q)) \cap M^d)^o$ is the open neighborhood on M^d as above, then $|pq| > |pq''|$, $\forall q'' \in ((\cup_{q\in Q} \bar{B}^n(q, r_q)) \cap M^d)^o$.

(4) In an exactly similar way we will call Q a set of constant infimum distance from M^d if Q, M^d and postulates (a), (b), (c) are as in (3) above, but in postulate (d) we have $|pq|<|pq''|$ with everything else in d as above.

Now we are equipped to state our next corollary, based on theorem (2-5). It has to do with curves $M^1 \subset R^n$, where M^1 has all the properties mentioned in the previous theorems and corollary (3.1).

***3.3.* Theorem** *A necessary and sufficient condition for p $\in R^n$ to belong to $\Pi(M^1)$ is that there exists a $Q \subset M^1$ which*

a) disconnects M^1
b) is of constant supremum distance from p.

Proof: A simple application of theorem (2-5) on M^1 if we notice that Q is necessarily closed in M^1 and since M^1 is connected and $M^1 \setminus Q$ is not, Q is a set of maximum constant distance from p.
Obviously Q can be any proper closed subset of a circle $S^1(p,r)$ with $r=|pq|$ $\forall q\in Q$. It is only natural, though, to see the smallest set Q which will be necessary and sufficient so that $p\in \Pi(M^1)$. If Q disconnects M^1, then since M^1 is homeomorphic to a one-dimensional connected subset of R that is, M^1 is homeomorphic to either R, or $R_+ =[0,+\infty)$ or [0,1] then nay connected component of Q will disconnect M^1. Therefore

3.4. Corollary *The set Q of theorem (3.3) can be made to contain either a single point {q} or a proper connected arc of S^1 (p,|pq|).*

This last theorem (3.3) and corollary (3.4) were specific to curves $M^1 \subset R^n$. But their statements are made in such a general way that they don't take into account the fact that N is differentiable of order at least C^1. We can see that if $Q\subset M^1$ of theorem (3.3) is a set of constant maximum distance from p, then $\forall q\in Q$ either q is a boundary point of M^1 or |pq| is normal to M^1 at q. That is, if $q\in Q$ and q is not a boundary point of M^1, then $(|\overline{pq}|\cdot \bar{t}_q) = 0$, where $\bar{t}_q$ is the unit tangent vector of M^1 at q. But a necessary and sufficient condition for q not to be a boundary point of M^1 is that q disconnects M^1. Also a necessary and sufficient condition for any $q\in Q$ not to be a boundary point of M^1 is that Q disconnects M^1. So we can state a simple lemma which will help with our next theorem.

***3.5.* Lemma** *if M^1,p,Q are as in theorem (3.3) and corollary (3.4) then $\forall q \in Q$,*

$(|\bar{pq}|\cdot \bar{t}_q) = 0$. *That is, $|\bar{pq}|$ is normal to M^1 at q.*

This result is obvious, not only for a set of maximum distance from p, but also for a set of constant minimum distance, provided that it too does not contain boundary points of M^1. Therefore if we have a point $p \in IR^n$ and $M^1\subset IR^n$ is a curve as above, then in order to prove that $0 < K^{(1)}_{l.c.t}(p,M) < \infty$ and also to calculate it as a specific real positive number all we have to do is find the sets of maximum and infimum distance from p. If M^1 has no boundary points, then we can simply find all points $q_i \in M^1$ for which

$(|\bar{pq}|\cdot \bar{t}_4) = 0$.

Then we examine them for maxima or minima, dropping those for which |pq| is neither maximum nor minimum but designates a "point of inflection". One maximum distance |pq| is enough to prove that p$\in \Pi(M^1)$. But, to calculate $K^{(1)}_{l.c.t}$ (p,M^1) we need all the minima and infima. Next theorem explains how it is done.

***3.6.* Theorem** *Let $M^1 \subset IR^n$ be a curve as above and $p \in IR^n$. Then we have the following:*

(1) A necessary and sufficient condition for p $\in \Pi(M^1)$ is that there is a q $\in$ M^1 with the properties

(a) q $\in$Q where Q is a connected set of constant maximum distance and Q disconnects M^1.

(b) $(|\bar{pq}| \cdot \bar{t}_q) = 0$

(2) In order to calculate $K^{(1)}_{l.c.t}$ (p,M^1) we consider two cases:

(a) Let M^1 not have any boundary points. Then we find all sets Q of constant minimum distance from p and we choose exactly one q for every Q which is connected and contains either a single point q_j or an arc of constant radius $|pq_j|$, with p as a centre, as described in corollary (3.4). When then arrange for all j's the distances $|pq_j|$ in increasing order of magnitude: $|\bar{pq}_1 \leq |\bar{pq}_2 \leq |\bar{pq}_3 \leq ... \leq |\bar{pq}_j \leq$

Then we have, for j=2 $K^{(1)}_{l.c.t.}(p, M^1) = \dfrac{1}{|\bar{pq}_2|}$

(b) If M^1 has boundary points, at most they will be two. We examine them to see if they belong to sets of infimum distance from p. If they do, we include them in the increasing series of the $|pq_j|$ along with the minima we found from the $(|\bar{pq}| \cdot \bar{t}_q) = 0$ *normality relation. If they do not belong to the sets of infimum distance, we simply ignore them and proceed as in (a).*

Proof: Part (1) is simply a restatement of theorem (3.3) along with corollary (3.4) and so needs no extra proof. What we need to observe here is that if one set of constant max distance Q exists as if specified in part (1) of this theorem, then two points q_1, $q_2 \in IR^n$ exist which (a) belong to sets of constant infimum distance from p, (b) either they are boundary points of M^1, or q_1, $q_2 \in M^1$ and i = 1, 2, $(|\bar{pq_i}| \cdot \bar{t}_q) = 0$, (c) they are boundary points or belong to different connected components of M^1\Q. Part (a) of this statement is due to the fact that $\forall$q'$\in M^1$ we have that 0 ≤ |pq'|. So |pq'| is bounded below. Now if q$\in$Q, then, since Q is a set of constant maximum distance from p, if q' moves away from Q, |pq'| will be decreasing at least while q'$\in$ N(Q)\Q, where N(Q) an open neighborhood of Q as specified in Def. (2.4). As long as |pq'| is decreasing, it will fulfill the relation: 0 ≤ |pq'| ≤ |pq|, where q$\in$Q. Therefore |pq| will have an infimum, before it starts increasing again for its next maximum value. Part (b) has been explained above and, part (c) is a corollary of the fact that Q disconnects M, so we can have two points q_1', q_2' moving away from Q, therefore we have two points q_1', q_2' $\in M^1$\Q in different connected components for which $|pq_1|$ and $|pq_2|$ are infimum distances, i.e.,

they belong to sets Q_1, Q_2 of constant infimum distance from p which are included to different connected components of $M^1 \backslash Q$.
On the other hand and by similar reasoning we can see that if Q_1, Q_2 are two disjoint sets of constant infimum distance, which are successive, that is, there is not a third set of minimum distance Q' between them, so that Q' would disconnect M^1 and Q_1, Q_2 would belong to different connected components of $M^1 \backslash Q'$, then there is a set Q_3 of maximum distance from p, which is between Q_1 and Q_2, that is, it disconnects M^1 so that Q_1, Q_2 are included to different connected components. The proof of this is based on the fact that if $q_1 \in Q_1$ and $q_2 \in Q_2$, the arc (q_1, q_2) which connects them on M^1 is homeomorphic to [0,1], so it is compact. Therefore, if $|pq|$ is a real non negative function of $q \in [q_1, q_2]$, the set of values of $|pq|$ is also compact in R, so it has a maximum value. The set Q_3 of all points, $q_3 \in [q_1, q_2]$, for which $|pq_3|$ has this maximum value, is connected, otherwise we would have sets of minimum value among the maxima, which has already been excluded. So, Q_3 either contains a single point q_3 or is an arc of a circle with p as a centre and the maximum value as a radius. That is Q_3 is a set of constant maximum value as in Def. (3.2).
Therefore to prove the second part of the theorem, we consider the first two sets Q_1, Q_2 of infimum distance, for which $|p\bar{q}_1| \leq |p\bar{q}_2|$. Between them there will exist a set Q_3 of constant maximum distance. If now we define the section: $T = (\overset{o}{B}(p, \overline{|p\bar{q}_2|}) \cap M^1)$ we will have $Q_1, Q_2, \subset T$, $Q_3 \not\subset T$ since $\forall\, q_3 \in Q_3$ $\overline{|pq_3|}$ is a maximum, therefore $|p\bar{q}_2| < |p\bar{q}_3|$ and T is not connected, since Q_3 disconnects M^1. Also if $T' = B(p, \bar{r}) \cap M^1)$ where $r < |pq_2|$, then, since $|p\bar{q}_1| < |p\bar{q}_2|$ it may be that $|p\bar{q}_1| < r < |p\bar{q}_2|$ and $Q_1 \subset T'$ but $Q_2 \not\subset T'$ and then T' is connected. If it were non connected, then the connected component of T' which would not contain Q_1 would give us another minimum point $q_2' \notin Q$ and therefore the sequence of minimum values would be, by $\overline{|pq_2'|} \leq r \leq \overline{|pq_2|}$, $\overline{||pq_1|} \leq \overline{|pq_2'|} \leq \overline{|pq_2|} \leq \ldots$ which cannot be since $|pq_2|$ is the second term of the sequence, as given. By proving that T' is connected $\forall r < |pq_2|$, and that T is not connected for $r = |pq_2|$, we have proven that $K^{(1)}_{l.c.t.}(p, M^1) = \frac{1}{|pq_2|}$ and that concludes the proof of theorem (3.6).
We can now give the equivalent results for case (b), where M^1 is homeomorphic to S^1. In this case, M^1 has no boundary points. Also, one point q or one closed, connected subset Q of M^1 does not disconnect it. Things are similar in everything else, and also definitions (3.2), (1) and (2) are equally valid here. Definitions (3.2) (3) and (4) are not even needed, since the absence of boundary points and the compactness of M^1 makes suprema into maxima and infima into minima.
So we can immediately proceed to state some theorems analogous to those of case (a).

3.7. Corollary *Let $M^1 \subset R$, be a curve homeomorphic to S^1 and having all the other properties we assumed for the M^d 's. Then, a necessary and sufficient condition for a point $p \in R^n$ to belong to $\Pi(M^1)$ is the following: There are four successive points $q_1, q_3, q_2, q_4, \in M^1$. By successive we mean that if we subtract $\{q_1\}$ from M^1, then $M^1 \setminus \{q_1\}$ is disconnected by q_3, and q_2, q_4 belong to different connected components of $M^1 \setminus \{q_1, q_3\}$. Or, if we start from q_3 we find that q_1, q_2 belong to different connected components of $M^1 \setminus \{q_3, q_4\}$, etc. For these four points, the following relation holds: $\max\{|\overline{pq_1}|, |\overline{pq_2}|\} < \min\{|\overline{pq_3}|, |\overline{pq_4}|\}$.*

Proof: Again, this is an application of theorem (2.1), where Q = $\{q_3, q_4\}$ and q_1, q_2 play the same role as in theorem (2.1).
The next theorem, analogous to theorem (3.3) is the following:

3.8. Theorem *If $p \in R^n$ and $M^1 \subset IR^n$ as above, then a necessary and sufficient condition so that $p \in \Pi(M^1)$ is that there exist two disjoint subsets $Q_1, Q_2 \subset M^1$ which:*

(a) Have a union $Q = Q_1 \cup Q_2$ which disconnects M^1.
(b) Are of constant maximum distance from p.

Proof: First we chose that set of the two, Q_1 or Q_2 which has the greatest distance from p. Let it be Q_1. Then on $M^{1'} = M^1 \setminus Q_1$ we apply theorem (2.5), using Q_2 as Q. Any $\overline{B}(p,r)$ which will not intersect Q_1 will not intersect Q_2 also. This concludes the proof. In the same spirit, we have a corollary analogous to corollary (3.4).

3.9. Corollary *The sets Q_1, Q_2 of the previous theorem can be made to contain each either a single point, q_3 or q_4 or a proper connected arc of $S^1(p, |\overline{pq_4}|)$, as the case may be.*

Like corollary (3.4) the proof is superfluous
Considering that by definition of its properties, M^1 is differentiable of order C^1, we can give now a lemma which is analogous to lemma (3.5).

3.10. Lemma *If $p \in IR^n$, $M^1 \subset IR^n$ as above and Q_1, Q_2 are as in theorem (3.8) and corollary (3.9), then $\forall q \in Q_1 \cup Q_2$, we have:*

$$(|\overline{pq}| \cdot \overline{t_q}) = 0.$$

Normality condition. That is, $|\overline{pq}|$ is normal to M^1 at q. Again, no proof is needed.
We conclude our work with a final theorem, analogous to theorem (3.6):

3.11. Theorem *Let M^1 be a curve as above and If $p \in IR^n$. Then we have the following:*

(1) A necessary and sufficient condition for a point $p \in IR^n$ to belong to $\Pi(M^1)$ is that there exist $q_1, q_2, \in M^1$ with the properties:

(a) $q_1 \in Q_1$, $q_2 \in Q_2$ *, where* Q_1, Q_2 *are connected disjoint sets of constant maximum distance, and their union* $Q = Q_1 \cup Q_2$, *disconnects* M^1, Q_1 *disconnects* $M^1 \setminus Q_2$, *and* Q_2 *disconnects* $M^1 \setminus Q_1$.

(b) $\forall q_1 \in Q_1$, $\forall q_2 \in Q_2$, $(|\overline{pq_1}| \cdot \overline{t_{q_1}}) = (|\overline{pq_2}| \cdot \overline{t_{q_2}}) = 0$

(2) For calculating $K^{(1)}_{l.c.t.}(p, M^1)$ *, again as in theorem (3.6), we find all the* Q_j, *connected sets of minimum distance and, by selecting exactly one* q_j *for every* Q_j *we create the sequence of minimum distances in increasing order:*

$$|\bar{pq_1}| \le |\bar{pq_2}| \le |\bar{pq_3}| \le \ldots \le |\bar{pq_j}| \le \ldots.$$

So, $K^{(1)}_{l.c.t.}(p, M^1) = \dfrac{1}{|\overline{\overline{pq_2}}|}$.

Proof: Part (1) is obvious. For Part (2) we only have to notice that M^1, homeomorphic to S^1, is doubly connected. Therefore, two sets of minimum distance Q_1, Q_2 define two sets of maximum distance Q_3, Q_4 between them, one for each path connecting Q_1 to Q_2.

4. Conclusion This work is a continuation of previous papers [2] and [3]. The concepts are generalized and some theorems are given by which one can actually calculate $K^{(1)}_{l.c.t}$ and define $\Pi(M^d)$ for a given $M^d \subset R^n$. (For ordinary curves and surfaces, M^1 or M^2 in R^3, one can even make use of appropriate computer programs).

The usefulness of these new concepts can be seen from the fact that they are neither exclusively local nor exclusively global concepts. One can call $K^{(1)}_{l.c.t}$ and $\Pi(M^d)$ a local-global concepts [2],[3]. This fact, along with their ability to discern between surfaces of zero Gaussian curvature, can make them useful in the classification of curves and surfaces in an entirely new way. So these new concepts can be used to help classify medical graphs of various kinds, or help in the classification of surfaces, in problems of General Relativity, [3]. We think that they might prove useful in other branches of mathematics, as in catastrophe theory, or in problems in physics, where fractal sets, or attractors in chaos situations are involved.

References

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