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Curl Eigen-fields and Gauge-like Modes of Linear Wave Equations

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Abstract. A general overview of the usefulness of Beltrami - Trkal Vortical Field Theory on physics is reviewed and an as yet unnoticed possible extention is presented. The author provides evidence that a whole new class can be found through two scalar gauge potentials with possible connections with a "‘hidden sector"’ in both electrodynamics and Maxwellian electromagnetism associated with the recently introduced Meta-fluid dynamics.

1 Introduction

The notion of a force-free field comes directly from the work of 19th century mathematician Eugenio Beltrami (1835 - 1899) in hydrodynamics [1]. In fact Beltrami made a very important contribution by a direct comparison between electrodynamics and hydrodynamics probably inspired by Lord Kelvin's vortex model of the atom. The central notion is that of a generalized eigenvalue of a rotation operator. As this is allowed to vary also with space and time it is preferable to call it the eigen-vorticity and we use this term for the rest of this paper. The special case of constant eigen-vorticity is often met in the literature under the name of a Trkalian flow from the Czech physicist and mathematician Viktor Trkal (1888 -1955) who has done similar work independently back at 1919 [2]. The subject was proven to be of significance in fluid mechanics, electromagnetism, magnetohydrodynamics and astrophysics.

In the following, we briefly introduce the subject of Beltrami fields and their general significance for Maxwell theory in section 2. In section 3, we present a general treatment for the curl-eigenfield problem beyond Beltrami and show the existence of an additional class of paradoxical non-solenoidal solutions for the linear wave operator. In section 4, we derive an important "‘advection"’ equation as a constraining condition for non-solenoidal curl eigen-fields and we describe the solution through the introduction of an appropriate scalar potential. In section 5, we explore the possibility of spherically symmetric solutions of the non-solenoidal problem in terms of Vector Spherical Harmonics. In section 6, we further examine the content of such solutions through their connections with non-linear Euler hydrodynamics. In section 7, we conclude on the possibility of excitation of this new class of solutions and their possible new applications in engineering electromagnetics.

2 Review of Beltrami-Trkal theory and its applications

A curl eigen-field is one that satisfies the eigen-rotation equation

$$\nabla \times \mathbf{A}(\mathbf{r}, t) = \lambda(\mathbf{r}, t)\mathbf{A}(\mathbf{r}, t) \tag{1}$$

In the above, $\lambda(\mathbf{r}, t)$ is the eigen-vorticity (vorticity eigenvalue), a generalization of the notion of eigenvalue for the rotation operator. The possibility of different orientations has been absorbed in the scalar coefficient. The inverse operator is often given in terms of the "'Biot-Savart"' integral operator [3] well known from ordinary classical electrodynamics.

Such fields have been initially introduced in hydrodynamics to express vortex-like structures in connection with the stability of solutions of the Euler equation. The first time we meet again a similar term is at the 50's when Woltier [4], Lundquist [5] and Chandrasekhar-Kendall [6] introduced the notion of a "Force-Free" magnetic field, satisfying (1) but with λ constant, in order to describe the state of equilibrium of intergalactic plasma. It is then proven that in lower frequencies where one can neglect the displacement current, the Lagrangian of the magnetic part finds its natural minimum in such configurations where the current term becomes parallel to the magnetic field thus eliminating any Lorentz forces inside the plasma.

In the generic case of Magneto-hydrodynamics we usually start from the magnetic B field which can be derived through (1) and through Maxwell equations we can deduce the sources. On the other hand the second Maxwell equation for constant λ demands that both

$$\nabla \times \mathbf{B} = \lambda \mathbf{B} \tag{2}$$

$$\nabla \bullet \mathbf{B} = 0 \tag{3}$$

should hold which restricts possible solutions of Beltrami eigen-rotation equation. An additional constraint is derived later from the vanishing curl over (2).

A comprehensive review has been given by G. Marsh [7]. As is immediately obvious, application of the curl operator into (1) turns it to a non-linear dispersion vector wave or Helmholtz equation, which has been solved for constant λ and in some cases for varying eigenvalue $\lambda(\mathbf{r})$ with appropriate boundary conditions. Trkalian fields with $\lambda = \pm 2$ are known in the mathematical literature as Left Invariant Fields with the simplest cases given by $x_i \mathbf{e_i}$. Non-constant eigen-vorticity is much more difficult and it has been known for decades in solar physics and astrophysics through the Grad-Shafranov equation (see [7], chap. 3, p. 25-46) for which several special solutions have been studied extensively. Not many general solutions of (1)-(3) are known in the literature (see [10] - [18]). Recent extensions of previous solutions have also been proposed ([14] - [18] including the so-called Ball-like Force-Free fields ([14], [15]).

A similar notion has been extensively used in the study of bi-anisotropic media especially by A. Lakhtakia [8] and others. The later author has also shown in [9] that there exists a reformulation of Maxwell postulates in a purely covariant form in terms of two complementary complex Beltrami electromagnetic fields of the form

$$\mathbf{Q}_{\pm} = \frac{1}{2}[\mathbf{E} \pm \mathbf{i}Z\mathbf{B}] \tag{4}$$

where Z stand for the scalar vacuum impendance.

An alternative approach on non-constant solutions is based on the decomposition of solenoidal (divergence-free) vector fields in R3 in the form

$$\mathbf{A} = \nabla \times \phi \mathbf{e}_1 + \nabla \times \nabla \times \psi \mathbf{e}_2 \tag{5}$$

where ϕ and ψ are called the Debye-Hertz potentials also known as the Polloidal-Toroidal decomposition. This method has been employed by Benn and Kress in [10] to find solutions of equation (1) in terms of such potentials. There is also an orthogonal basis of complex curl eigenfunction modes and the relevant complex helical wave decomposition first introduced by Lesieur [19] who defined helical waves through

$$\mathbf{V}_{\pm} = [\mathbf{a}(\mathbf{r}) \pm \mathbf{i}\mathbf{b}(\mathbf{r})] \exp(\mathbf{ikr}) \tag{6}$$

The above is shown to satisfy $\nabla \times \mathbf{V}_{\pm} = \pm|\mathbf{k}|\mathbf{V}_{\pm}$. A similar orthogonal decomposition in Fourier space has been introduced by Moses [20].

A deeper treatment by Kravchenko in [21] based on differential forms calculus, has resulted in an equivalent set of three Schroedinger equations with a non-constant complex eigen-vorticity. It is a notable fact that both Benn and Kress and also Pantilie and Wood in [22] who have made a covariant treatment of (1) agree that the case of non-constant eigen-vorticity is inherently connected with curved metrics and the construction of self-dual metrics of the form

$$g = rh + r(dr + A^2) \tag{7}$$

In fact, it is proven in [21] that there exists a linear morphism which maps solutions of the wave equation to solutions of a generalized eigen-roation equation given as

$$*dA = \pm cA \tag{8}$$

where * stands for the Hodge dual operator. Furthermore, Kassandrov and Trishin show in [23] that there exist spinorial generalizations of the Beltrami operator that occur in the theory of Shear-Free Congruencies in which every Maxwell-like field becomes self-quantized.

There is also an associated work by H. Marmanis [24] who brought afore the old and forgotten subject of the analogy between hydrodynamics and electrodynamics in the effort to impose quantization conditions in turbulent flows. This direction of research was soon assimilated after 2000 in the new area called Meta-fluid Dynamics which is a form of Gauge field theory. Similar in concept are the findings of Saygili [25] in his attempt to construct a topologically massive abelian gauge field theory. In this construct abelian gauge potentials on Riemannian manifolds are Trkalian fields which define contact structures. The deep relation between contact structures and Beltrami fields has also arisen in the mathematical literature especially from the study of the "'ABC"' flows with unstable hyperbolic orbits by Arnold [26]. In the next section, we attempt to derive a more general classification of curl eigen-fields including also non-solenoidal fields with applications in Maxwell theory in both the Coulomb and Lorentz gauge.

3 Curl eigen-fields beyond Beltrami-Trkal

Ordinary Beltrami-Trkal theory only describes solenoidal fields that are combatible with the usual Coulomb gauge ($\nabla \bullet \mathbf{A} = 0$) when applied to the vector potential. It is also combatible with ordinary Maxwell equation for the magnetic field in the case of Force-Free configurations in the absence of monopoles. Both these conditions require the fields to be soleonoidal satisfying eq. (3). It is though entirely possible to describe an additional class of fields in the Lorentz gauge for which $\nabla \bullet \mathbf{A} = \partial_t \Phi$ where Φ stands for the electric scalar potential. This case has been unjustifiably omitted from existing literature, at least to our knowledge.

We also note that, when the eigen-field condition (1) is applied at the vector potential instead of the magnetic field, results in a set of parallel electric and magnetic components satisfying $\mathbf{E} \times \mathbf{B} = 0$ and thus they cannot constitute ordinary radiation but they can be set up as stationary modes inside closed cavities. This was experimentally proven for ordinary Trkalian fields made up of left and right polarized modes inside a laser cavity by Le Flock et al. [27] Some controversy that had been stirred due to different approaches on this issue which seeemed to have been settled down after a final study by Gray in [28]. Nevertheless, the same issues may turn out to be slightly different in the second class that we examine here.

Both classes are characterized by an additional condition that comes out of the vector identity that requires the divergence of a curl to vanish. Application of this into eq. (1) is straightforward and leads ot the condition

$$\nabla\lambda \bullet \mathbf{A} + \lambda \nabla \bullet \mathbf{A} = 0 \tag{9}$$

This has then to be treated differently for each gauge used. We thus, complete the classification of curl eigen-fields problem by stating the two differently defined systems of PDEs.

3.1 Beltrami Class

Combining (1), the Coulomb gauge and (9) leads to the system of PDEs

$$\nabla \times \mathbf{A} = \lambda \mathbf{A} \tag{10}$$

$$\nabla\lambda \bullet \mathbf{A} = 0 \tag{11}$$

$$\nabla \bullet \mathbf{A} = 0 \tag{12}$$

Direct application of the general identity for the Laplacian

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \bullet \mathbf{A}) - \nabla^2 \mathbf{A} \tag{13}$$

and use of (10) leads to

$$\nabla \times (\lambda \mathbf{A}) = \nabla\lambda \times \mathbf{A} + \lambda^2 \mathbf{A} = -\nabla^2 \mathbf{A} \tag{14}$$

which allows to connect the above with the linear wave or Helmholtz eq. $[\nabla^2 - k^2]\mathbf{A} = \mathbf{J}$ (assuming monochromatic time dependence) with a "source" term of the form

$$\mathbf{J} = -\nabla\lambda \times \mathbf{A} - \Omega^2 \mathbf{A} \tag{15}$$

where $\Omega^2 = \lambda^2 + k^2$. We note that the derivation above, is not restricted to a constant wave vector and we can in principle also incorporate any scalar function $k(\mathbf{r})$ for an inhomogeneous medium.

3.2 Non-solenoidal Class

In the second class, application of (9) and the Lorentz gauge leads to the new problem

$$\nabla \times \mathbf{A} = \lambda \mathbf{A} \tag{16}$$

$$\nabla \bullet \mathbf{A} = -\nabla \log \lambda \bullet \mathbf{A} = \phi \tag{17}$$

where $\phi = \partial_t \Phi$ or $\pm \mathbf{i}\omega\Phi$ in case of single frequency. We also introduce for convenience the auxilliary vector field $\mathbf{u} = \lambda^{-1}\nabla\lambda = \nabla \log \lambda$ to write (16) as

$$\nabla \bullet \mathbf{A} = -\mathbf{u} \bullet \mathbf{A} = \phi \tag{18}$$

Again, the double curl identity for the Laplacian leads to

$$\nabla\lambda \times \mathbf{A} + \lambda^2 \mathbf{A} = \nabla\phi - \nabla^2 \mathbf{A} \tag{19}$$

which is also combatible with the wave or the associated (generally inhomogeneous) Helmholtz equation $[\nabla^2 - k^2]\mathbf{A} = \mathbf{J}$ this time with the source term

$$\mathbf{J} = \nabla\phi - \Omega^2 \mathbf{A} - \lambda\mathbf{u} \times \mathbf{A} \tag{20}$$

An important obervation concerns the possibility of cancellation of the source term. This is geometrically impossible in eq. (15) but it can be possible by an appropriate choice of the scalar potential term in (20). Hence, we deduce that if there exist solutions of the problem defined by the above field equations then there exist also a special subclass of non-solenoidal curl eigen-fields that can in principle satisfy the sourceless linear wave equation in vacuum or in an appropriate medium. This subclass is defined by the set of PDEs

$$\nabla \bullet \mathbf{A} = -\mathbf{u} \bullet \mathbf{A} = \phi \tag{21}$$

$$\nabla \times \mathbf{A} = \lambda \mathbf{A} \tag{22}$$

$$\nabla\phi - \Omega^2 \mathbf{A} - \lambda\mathbf{u} \times \mathbf{A} = 0 \tag{23}$$

In the next section we explore possible solutions and consistency of the above.

4 The general Double Advection Equation for the non-solenoidal class

The last set of equations allows to eliminate the scalar term ϕ via equation (21) which when introduced in (23) results in

$$\nabla(\mathbf{u} \bullet \mathbf{A}) + \lambda\mathbf{u} \times \mathbf{A} + \Omega^2 \mathbf{A} = 0 \tag{24}$$

In what follows we simplify notation by introducing the tensorial matrix representation of the exterior product $\lambda\mathbf{u} \times \mathbf{A} = \overline{R}_{\mathbf{u}} \bullet \mathbf{A}$ where the $\overline{R}_{\mathbf{u}}$ matrix is written as

$$\overline{R}_{\mathbf{u}} = \begin{bmatrix} 0 & -\partial_z\lambda & \partial_x\lambda \\ \partial_z\lambda & 0 & -\partial_y\lambda \\ -\partial_x\lambda & \partial_y\lambda & 0 \end{bmatrix} \tag{25}$$

Thus we may simplify (24) by rewriting it in the form

$$\nabla(\mathbf{u} \bullet \mathbf{A}) + \left(\overline{R}_{\mathbf{u}} + \Omega^2 \mathbf{I}\right) \mathbf{A} = 0 \tag{26}$$

We next perform the expansion of the first term according to the general vector identity for vector products as

$$\nabla(\mathbf{u} \bullet \mathbf{A}) = \mathbf{u} \bullet \nabla\mathbf{A} + \mathbf{A} \bullet \nabla\mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{u}) \tag{27}$$

We immediately observe that the last term cancels $(\nabla \times \nabla (\log\lambda))$ while the third term is of the form $\mathbf{u} \times \lambda\mathbf{A}$ which can be added to the matrix term. Then (26) becomes

$$\mathbf{u} \bullet \nabla\mathbf{A} + \mathbf{A} \bullet \nabla\mathbf{u} + \overline{M}\,(\lambda, k) \bullet \mathbf{A} = 0 \tag{28}$$

where we introduced the total matrix

$$\overline{M}\left(\lambda,k\right)=2\overline{R}_{\mathbf{u}}+\Omega^{2}\mathbf{I}=\begin{bmatrix}\lambda^{2}+k^{2} & -2\partial_{z}\lambda & 2\partial_{x}\lambda\\ 2\partial_{z}\lambda & \lambda^{2}+k^{2} & -2\partial_{y}\lambda\\ -2\partial_{x}\lambda & 2\partial_{y}\lambda & \lambda^{2}+k^{2}\end{bmatrix} \tag{29}$$

To decoenvolve the two vector fields we introduce the new composite field $\mathbf{F}=\mathbf{u}+\mathbf{A}$ and the associated advective derivative $\mathbf{F}\bullet\nabla\mathbf{F}$ that allows separating terms in (28) as

$$\mathbf{F}\bullet\nabla\mathbf{F}-\mathbf{u}\bullet\nabla\mathbf{u}-\mathbf{A}\bullet\nabla\mathbf{A}+\overline{M}\left(\mathbf{u};\lambda,k\right)\bullet\mathbf{A}=0 \tag{30}$$

At this point we may introduce a standard identity for advective derivatives in the form

$$\mathbf{u}\bullet\nabla\mathbf{u}=\nabla\left(\left\|\mathbf{u}\right\|^{2}/2\right)+\nabla\times\mathbf{u}\times\mathbf{u} \tag{31}$$

$$\mathbf{A}\bullet\nabla\mathbf{A}=\nabla\left(\left\|\mathbf{A}\right\|^{2}/2\right)+\nabla\times\mathbf{A}\times\mathbf{A} \tag{32}$$

of which the last terms vanish identically. The same expansion for the $\mathbf{F}$ derivative results in a term $\nabla\times\mathbf{F}\times\mathbf{F}$ of which only a single term of the form $\lambda\mathbf{A}\times\mathbf{u}$ survives which has to be subtracted from the original $\overline{M}$ matrix now being $\widetilde{M}=\overline{R}_{\mathbf{u}}+\Omega^{2}\mathbf{I}$. Then we get the final equation

$$\frac{1}{2}\nabla\left(\left\|\mathbf{u}+\mathbf{A}\right\|^{2}-\left\|\mathbf{u}\right\|^{2}-\left\|\mathbf{A}\right\|^{2}\right)+\widetilde{M}\bullet\mathbf{A}=0 \tag{33}$$

Having now eliminated the scalar from the initial equations (21 - 23) we may rewrite the problem definition in the form

$$\nabla\bullet\mathbf{A}=-\mathbf{u}\bullet\mathbf{A} \tag{34}$$

$$\nabla\times\mathbf{A}=\lambda\mathbf{A},\ \lambda=e^{\int_{\Omega}\mathbf{u}d\Omega} \tag{35}$$

$$\nabla^{2}\mathbf{u}=\rho=\lambda^{-1}\nabla^{2}\lambda-\lambda^{-2}|\nabla\lambda|^{2} \tag{36}$$

$$\nabla\times\mathbf{u}=0 \tag{37}$$

$$\nabla\left(\left\|\mathbf{u}\right\|^{2}\left\|\mathbf{A}\right\|^{2}\right)+\widetilde{M}\bullet\mathbf{A}=0 \tag{38}$$

The additional $\mathbf{u}$ field is similar to an "‘electrostatic"’ one but the coupling in (34) leads directly to hard non-linear field theory. According to Helmholtz theorem, given both the divergence and rotation of a field suffice to define it up to a scalar gauge which for the Lorentz gauge would require some additional boundary conditions on the light cone []. One may though restrict attention to special modes in waveguides and closed cavities given the paradoxical nature of parallelization of electric and magnetic components that occurs for this type of vector potentials [].

The above set of PDEs then, seems to contain an inifinite subclass of solutions for a vector potential, given an arbitrary irrotational $\mathbf{u}$ field. The most important step that remains is to deepen our understanding on the nature of the constraint given by equation (38). This may be seen as a nonlinear "‘Constitutive Relation"’ which contains an important piece of information on the generic characteristics of this particular subclass. This is revealed by taking the curl of (38) giving

$$\nabla\times\acute{\mathbf{A}}=0,\ \acute{\mathbf{A}}=\widetilde{M}\bullet\mathbf{A} \tag{39}$$

We then see that constraint (38) implies that the deformed vector potential $\mathbf{\acute{A}}$ must be irrotational. Correspondingly, the initial solution should be derivable from a transformation of the gradient of a scalar potential Π in the form

$$\mathbf{A} = \widetilde{M}^{-1}\nabla\Pi \tag{40}$$

Additional restrictions in the choice of the arbitrary scalar potential Π must take into account the full application of (38). Equation (40) still allows two degrees of freedom in the field definition. We can now plug our general form for $\mathbf{A}$ back into (38) for further analysis of the constraint. We immediately get

$$\nabla\left(\frac{1}{\lambda^2}\left\|\nabla\lambda\right\|^2\left\|\widetilde{M}^{-1}\nabla\Pi\right\|^2\right) - \nabla\Pi = 0 \tag{41}$$

Apparently, this can be integrated once to give an integration constant so that we end up with an equation reminiscent of the Hamilton-Jacobi class

$$\left\|\widetilde{M}^{-1}\nabla\Pi\right\|^2 - \left(\frac{\lambda^2}{\left\|\nabla\lambda\right\|^2}\right)\Pi = \epsilon \tag{42}$$

In fact, the above is practically identical with the classical Hamilton-Jacobi problem (non-linear version) given a curved metric (propagation inside an inhomogenous medium as in 1st order Eikonal approximation []) and an action-dependent potential $V(\lambda, \Pi)$. It might though be recognised as reminiscent of a classical analogue to the *Schroedinger* operator if we write it in the form

$$(\nabla\Pi\widetilde{M}^{-1})^T(\widetilde{M}^{-1}\nabla\Pi) + V(\mathbf{r})\Pi = \epsilon \tag{43}$$

$$V(\mathbf{r}) = -\left(\frac{\lambda}{\nabla\lambda}\right)^2 \tag{44}$$

We identified derivatives over new coordinates with the Jcobian $\widetilde{M}^{-1}$ and the associated mass-metric tensor

$$g = \left(\widetilde{M}^{-1}\right)^T \widetilde{M}^{-1} \tag{45}$$

It is enlightening to rewrite the initial field equations (34)-(35) in the new form

$$\nabla \bullet \mathbf{A} = -\lambda^{-1}\nabla\lambda \bullet \widetilde{M}^{-1}\nabla\Pi \tag{46}$$

$$\nabla \times \mathbf{A} = \lambda\widetilde{M}^{-1}\nabla\Pi \tag{47}$$

Thus the problem can be redefined as finding two scalar potentials (λ, Π) simultaneously satisfying (44) together with (40) and (43) together with (46)-(47). We note in passing that the problem of the metric elements defined by λ being associated with Π directly leads to a difficult self-interacting field theory with a potential of the form $-\Pi^3\left|\nabla\Pi\right|^{-2}$ and a Π-dependent metric tensor for which we have no evidence of a possible solution although it might find some application in new cosmological approaches on gravity and the dark energy/dark matter problems.

5 Existence of special spherical solutions for the non-solenoidal problem

In a previous report [34], Papageorgiou and Raptis attempted an approximation of the spherical curl eigen-fields without examining the gauge problem, via a Transmission Line technique. We will now show that for the Lorentz gauge there can be no spherically symmetric solution of the second class of section **3.2**. Keeping the same notation as in [34], we may write the VSH expansion of the vector potential together with (34) and (35) as

$$\mathbf{A} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[a_{lm}(r)\boldsymbol{\Upsilon}_{lm} + b_{lm}(r)\boldsymbol{\Psi}_{lm} + c_{lm}(r)\varPhi_{lm}\right] \tag{48}$$

$$\nabla\mathbf{A} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\left(\dot{a}_{lm} + \frac{2}{r}a_{lm}\right) - \frac{L^2}{r}b_{lm}\right]\boldsymbol{\Psi}_{lm} \tag{49}$$

$$\nabla\times\mathbf{A} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[-\frac{L^2}{r}c_{lm}\boldsymbol{\Upsilon}_{lm} - \left(\dot{c}_{lm} + \frac{c_{lm}}{r}\right)\boldsymbol{\Psi}_{lm} + \left(\dot{b}_{lm} + \frac{b_{lm}}{r} - \frac{a_{lm}}{r}\right)\varPhi_{lm}\right] \tag{50}$$

In the above we use the abbreviation $L^2 = l(l+1)$. Taking into account the irrotational character of the arbitrary driving field $\mathbf{u}$ from eq. (36), we may write its expansion in terms of the gradient of a scalar as

$$\mathbf{u} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\dot{\varLambda}_{lm}\boldsymbol{\Upsilon}_{lm} + \frac{\varLambda_{lm}}{r}\boldsymbol{\Psi}_{lm}\right] \tag{51}$$

The new scalar for the inner product $\phi = -\mathbf{u}\bullet\mathbf{A}$ has then the expansion

$$\phi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\dot{\varLambda}_{lm}a_{lm} + \varLambda_{lm}b_{lm}\right] = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\phi_{lm}\varPsi_{lm} \tag{52}$$

For the rest fo the proof we will not need the explicit form of the expansion coefficients for ϕ_{lm} so we simply plug (49) and (52) into (34) to get

$$\dot{a}_{lm} + \frac{2}{r}a_{lm} - \frac{L^2}{r}b_{lm} = \phi_{lm} \tag{53}$$

From (35) we get the additional conditions

$$-\frac{L^2}{r}c_{lm} - \lambda a_{lm} = 0 \tag{54}$$

$$\dot{c}_{lm} + \frac{1}{r}c_{lm} + \lambda b_{lm} = 0 \tag{55}$$

$$\dot{b}_{lm} + \frac{1}{r}\left(b_{lm} + a_{lm}\right) - \lambda c_{lm} = 0 \tag{56}$$

We have used $\lambda = \exp\int_{\varOmega}\mathbf{u}d\varOmega$, where $\varOmega$ any bounded domain. Summing up (53), (59) and (56) we end up with the ODE system

$$\dot{\mathbf{v}} = -\frac{1}{r}\mathbf{W}\bullet\mathbf{v} + \mathbf{f} \tag{57}$$

where $\mathbf{v} = [a_{lm}, b_{lm}, c_{lm}]$, $\mathbf{f} = [\phi_{lm}, 0, 0]$ and the system matrix is of the form

$$\mathbf{W} = \begin{bmatrix} 2 & -L^2 & 0 \\ 1 & 1 & -\lambda r \\ 0 & \lambda r & 1 \end{bmatrix} \tag{58}$$

An additional constraint given by (54) is also written as the projection

$$\mathbf{t}\mathbf{v} = 0 \tag{59}$$

where the constraint coefficients are given by $\mathbf{t} = \left[\lambda, 0, \frac{L^2}{r}\right]$. Differentiating this gives

$$\dot{\mathbf{t}}\mathbf{v} + \mathbf{t}\dot{\mathbf{v}} = \dot{\mathbf{t}}\mathbf{v} + \mathbf{t}\mathbf{W}\mathbf{v} \tag{60}$$

The first part can also be incorporated to the square form if we introduce a new matrix $\mathbf{K}$

$$\mathbf{K} = \begin{bmatrix} \frac{\dot{\lambda}}{\lambda} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{r^2} \end{bmatrix} \tag{61}$$

such that (60) becomes

$$\mathbf{t}\left(\mathbf{K} + \mathbf{W}\right)\mathbf{v} = 0 \tag{62}$$

We now show that (59) and (62) cannot simoultaneously be combatible with a general solution of (57). For a linear system of ODEs, such a solution is provided by the exponential matrix over $\frac{-1}{r}\mathbf{W}$. On the other hand, (59) implies an orthogonality of the flow of (57) with the (a_{lm}, c_{lm})-plane. That is, the flow of (57) must be strictly oriented towards the b_{lm} axis. This though, is impossible given the mixing of subspaces by the matrix elements of (58). From the generic nature of the field problem defined by (34) - (37), we conclude that if there are general solutions there will probably fall into the class of toroidals or more general helical-spiral topologies.

Nevertheless, one can find some special spherically symmetric solutions that are compatible with the wave equation with sources. We show a simple example that could prove useful in experimental research. One starts with a pair of *dual* fields $\mathbf{A}_1, \mathbf{A}_2$ such that

$$\nabla \times \mathbf{A}_1 = \lambda_1 \mathbf{A}_2 \tag{63}$$

$$\nabla \times \mathbf{A}_2 = \lambda_1 \mathbf{A}_1 \tag{64}$$

Then, a solution of the full eigen-field problem can be constructed directly through a linear combination of the above. From standard vector identities for VSH one has

$$\nabla \times f(\mathbf{r})\boldsymbol{\Upsilon}_{lm} = -\frac{f(\mathbf{r})}{r}\boldsymbol{\Phi}_{lm} \tag{65}$$

$$\nabla \times f(\mathbf{r})\boldsymbol{\Phi}_{lm} = -\frac{L^2}{r}f(\mathbf{r})\boldsymbol{\Upsilon}_{lm} - \left(\dot{f} + \frac{f}{r}\right)\boldsymbol{\Psi}_{lm} \tag{66}$$

The choice $f(\mathbf{r}) = k/r$ eliminates the second term in the *rhs* of (66) thus allowing to write the dual fields as

$$\mathbf{A}_1 = \frac{k}{r}\mathbf{\Upsilon}_{lm}, \;\; \mathbf{A}_2 = \frac{k}{r}\mathbf{\Phi}_{lm} \tag{67}$$

For these, (63)-(64) are satisfied with $\lambda_1 = -1/r, \lambda_2 = -L^2/r$. In order to find a symmetric linear combination of these fields we assume a set of coefficients that depend on L^2 and write the total field in the form

$$\mathbf{A} = L^{2\mu}\mathbf{A}_1 + L^{2\nu}\mathbf{A}_2 \tag{68}$$

The two unkown exponents are to be found so that the eigen-rotation equation to be satisfied by the particular linear combination. Applying the curl we observe that coefficients get interchanged so that

$$\nabla \times \mathbf{A} = \lambda_1 \left(L^{2(\nu+1)}\mathbf{A}_1 + L^{2\mu}\mathbf{A}_2 \right) \tag{69}$$

For the final combination to be written in proportion with the original in (69) we must then take

$$\nabla \times \mathbf{A} = \lambda_1 \left(L^{2(\nu-\mu+1)}L^{2\mu}\mathbf{A}_1 + L^{2(\mu-\nu)L^{2\nu}\mathbf{A}_2} \right) \tag{70}$$

In order to get a common factor, the following condition must be satisfied

$$L^{2(\nu-\mu+1)} = L^{2(\mu-\nu)} \Leftrightarrow L^{4(\nu-\mu)+2} = 1 \tag{71}$$

This is equivalent to the equation $2(\nu-\mu)+1 = 0$ or $\mu = \nu + 1/2, \;\nu = 0, 1, 2, ...$ Then from (70) we find the eigen-vorticity as $\lambda = \lambda_1 L = -L/r$. Thus we obtain the following special eigenfields.

$$\mathbf{A}^{(\nu)} = \frac{k}{r}\left(L^{2\nu+1}\mathbf{\Upsilon}_{lm} + L^{2\nu}\mathbf{\Phi}_{lm} \right) \tag{72}$$

The auxilliary field $\mathbf{u}$ becomes then the unit radial $\tilde{\mathbf{r}}$ and the divergence scalar

$$\nabla \bullet \mathbf{A} = -\frac{L^{2\nu+1}}{r}\Upsilon_{lm} \tag{73}$$

In the next section we attempt a different approach to the combined problem of a solution based on the new scalar potentials introduced in (40) for the pure vaccum case.

6 Vacuum gauge fields through non-linear Euler hydrodynamics

Attacking directly the double advection equation together with the field equations for the $(\mathbf{u}, \mathbf{A})$ pair, might be a very complex problem due to the non-linearities involved. Nevertheless, the analysis of section **4** is suggestive for the hidden connection of (28) with the fully non-linear Euler hydrodynamics. From the generic decomposition given in (30) we see that one can introduce two separate ideal fluids each satisfying

$$\partial_t\mathbf{u} + \mathbf{u} \bullet \nabla\mathbf{u} = -\nabla p_1 \tag{74}$$
$$\partial_t\mathbf{A} + \mathbf{A} \bullet \nabla\mathbf{A} = -\nabla p_2 \tag{75}$$

While we used two hypothetical pressure terms p_1, p_2 they will be removed at the final result. Then, from the general identity

$$\mathbf{F} \bullet \nabla \mathbf{F} = (\mathbf{u} \bullet \nabla \mathbf{u} + \mathbf{A} \bullet \nabla \mathbf{A}) + (\mathbf{u} \bullet \nabla \mathbf{A} + \mathbf{A} \bullet \nabla \mathbf{u}) \tag{76}$$

and using (74) and (75) for the first part in *rhs* and (28) for the second part we finally get

$$\mathbf{F} \bullet \nabla \mathbf{F} = (-\partial_t \mathbf{F} + \nabla(p_1 + p_2)) - \widetilde{\mathbf{M}} \bullet \mathbf{A} \tag{77}$$

We conclude that the total field $\mathbf{F} = \mathbf{u} + \mathbf{A}$ must necessarilly satisfy

$$\partial_t \mathbf{F} + \mathbf{F} \bullet \nabla \mathbf{F} = -\nabla P - \widetilde{\mathbf{M}} \bullet \mathbf{A} \tag{78}$$

where $P = p_1 + p_2$.Then, using (40) we get

$$\partial_t \mathbf{F} + \mathbf{F} \bullet \nabla \mathbf{F} = -\nabla \left(P + \Pi\right) \tag{79}$$

One now observes that even in the abscense of any initial pressure terms, the scalar potential Π acts like a pressure term by itself on the corresponding Euler equation for the total field $\mathbf{F}$. Recalling that $\mathbf{u}$ is actually a gauge term of the form $\mathbf{F} = \mathbf{A} + \nabla(\log \lambda)$ we conclude that Π is essentially a *gauge pressure*.

From the above we can also derive the equivalent field theory

$$\nabla \bullet \mathbf{F} = \rho - \mathbf{u} \bullet \mathbf{F} + |\mathbf{u}|^2 \tag{80}$$

$$\nabla \times \mathbf{F} = \lambda \mathbf{F} - \nabla \lambda \tag{81}$$

$$\partial_t \mathbf{F} + \mathbf{F} \bullet \nabla \mathbf{F} = -\nabla \Pi \tag{82}$$

$$\mathbf{u} = \nabla(\log \lambda), \rho = \lambda^{-1} \nabla^2 \lambda - \lambda^{-2} |\nabla \lambda|^2 \tag{83}$$

In the above theory, the total field is uniquely described by the two scalar potentials λ, Π as

$$\mathbf{F} = \frac{1}{\lambda} \nabla \lambda + \widetilde{\mathbf{M}}^{-1}(\lambda, k) \nabla \Pi \tag{84}$$

We notice that the above can also be symmetrized using appropriate projectors in the form

$$\mathbf{F} = \mathbf{P}_+ \nabla \Pi_+ - \mathbf{P}_- \nabla \Pi_- \tag{85}$$

$$\mathbf{P}_\pm = \frac{1}{2} \left(\widetilde{M}^{-1} \pm \mathbf{I} \right) \tag{86}$$

$$\Pi_\pm = \log \lambda \pm \Pi \tag{87}$$

It remains to question the validity of $\mathbf{F}$ as a vacuum vector potential in the Lorentz gauge in relation with the associated electric and magnetic components.

It is first assumed that $\mathbf{F}$ satisfies

$$\nabla^2 \mathbf{F} - \partial_t^2 \mathbf{F} = 0 \tag{88}$$

$$\nabla \bullet \mathbf{F} - \partial_t \Phi = 0 \tag{89}$$

$$\nabla^2 \Phi - \partial_t^2 \Phi = 0 \tag{90}$$

Assuming an arbitrary dispersion we may replace the above with

$$\nabla^2 \mathbf{F} = \kappa^2 \mathbf{F} \tag{91}$$
$$\nabla \bullet \mathbf{F} = \kappa \Phi \tag{92}$$
$$\nabla^2 \Phi = \kappa^2 \Phi \tag{93}$$

Last one can be written explicitly using (83) and (84) as

$$\kappa \Phi = \nabla^2 \lambda - \nabla \lambda \bullet \mathbf{F} = \nabla^2 \lambda - \frac{1}{\lambda} |\nabla \lambda|^2 - \nabla \lambda \widetilde{\mathbf{M}}^{-1}(\lambda, k) \nabla \Pi \tag{94}$$

The question then is if there are matrices $\widetilde{\mathbf{M}}(\lambda)$ or $\mathbf{P}_{\pm}(\lambda)$ for which either of (84) and (85) is a valid representation for solutions of (91) and (82).

In order to explicitely solve for the two scalars we first choose to work with the symmetrized version (85) for $\mathbf{F}$ and then we use the following analysis for the transformed fields

$$\mathbf{F} = \sum_{i=1}^{3} f_i^+(\mathbf{r}) \hat{\mathbf{e}}_i - \sum_{i=1}^{3} f_i^-(\mathbf{r}) \hat{\mathbf{e}}_i \tag{95}$$

In the above decomposition each scalar f_i is associated with the corresponding row of the projection matrices $\mathbf{P}_{\pm}$ as

$$f_i^{\pm} = \mathbf{p}_i^{\pm} \bullet \nabla \Pi \tag{96}$$

Then the action of the Laplacian on $\mathbf{F}$ follows as

$$\nabla^2 \mathbf{F} - \kappa \mathbf{F} = \sum_{i=1}^{3} \left[\left(\nabla_i^2 f_i^+ - \kappa f_i^+ \right) - \left(\nabla_i^2 f_i^- - f_i^- \right) \right] \hat{\mathbf{e}}_i = 0 \tag{97}$$

At this point, we may introduce an additive separation of variables in the form

$$\Pi_{\pm} = \Pi_x^{\pm} + \Pi_y^{\pm} + \Pi_z^{\pm} \tag{98}$$

This effectively turns (96) as well as (81) into a set of six equations over six free parameters that in principle should be sovable with standard analytical or numerical techniques. Electric and magnetic fields should then be derived as usual by their defining relations as

$$\mathbf{E} = \nabla \Phi - \partial_t \mathbf{F} = \nabla(\Phi + \Pi) - \mathbf{F} \bullet \nabla \mathbf{F} \tag{99}$$
$$\mathbf{B} = \nabla \times \mathbf{F} = \lambda \mathbf{F} - \nabla \lambda \tag{100}$$

In this field theory, we see that we can find vacuum electric and magnetic components if there is a triplet of functions λ, Φ, Π such that

$$\nabla \bullet \mathbf{E} = \nabla^2(\Phi + \Pi) - \nabla \left(\mathbf{F} \bullet \nabla \mathbf{F} \right) = 0 \tag{101}$$
$$\nabla \bullet \mathbf{B} = \nabla \left(\lambda \mathbf{F} \right) - \nabla^2 \lambda = 0 \tag{102}$$

One should notice that the gauge pressure term Π appears to have same units with the longtitudinal electrostatic term Φ. Thus in a sense, the vector potential appears here to be a deformation of a longtitudinal gauge term.

7 Discussion and conclusions

The works mentioned in section **2** as well as the evidence presented here imply the possibility of alternative routes in Electrodynamics and Electromagnetism. It seems that a whole range of intriniscally non-linear fields can be derived by the extended class of curl eigenfields that are in principle compatible with Maxwellian field theory. The extention has been shown to occure out of the cross-section of two null spaces that can be simultanously satisfied, one of the ordinary linear wave or Helmholtz operator (albeight with generally nonlinear dispersion possible) and the other given by the peculiar source term (20) that allows for slef-elimination and possibly of self-quantization if taken inside appropriate boundaries.

Such an extended class has been proven to be associable with a set of underlying scalar potentials that might support localized solutions as interference patterns that could have a deep connection with some old and recent electromagnetic mass theories. It should be noted with great caution, that while in the final expression (85) the total gauge fields were symmetrized using a real linear combination, we could have also chosen a complex combination. Then, the inherent spinorial symmetry would suggest the co-existence of two complementary fields of the form

$$\mathbf{F}^{\pm} = \mathbf{P}_{+}\nabla\Pi_{+} \pm Z\mathbf{i}\mathbf{P}_{-}\nabla\Pi_{-} \tag{103}$$

where $\mathbf{P}_{\pm}$ appropriate complex projectors. Would this mean that photon polarization already existed inside a hidden sector of Maxwell's theory? We postpone answers to these and other questions for a later more detailed study.

It is notable that the geons hypothesis first proposed by Wheeler [29] and later elaborated by Melvin [30] and others could have stable solutions in this new class. It could also be contrasted with certain renewed interest in Reinich's "'Already Unified"'field theory [31], [32] and geometrodynamics [33] in general. Based on this evidence we find reasons to believe that the opposite route is also possible through which an understanding of both Maxwell and Yang-Mills fields could be reduced to an appropriate generalization of Navier-Stokes equations describing a self-quantizing, relativistic superfluid structure of a primordial, non-linear vacuum. Moreover, as far as a relativistic, covariant description of hydrodynamics exists it seems possible to include all Maxwell-like fields into a unified description where such forces will turn out to be inertial in nature. Meta-fluid dynamics, although in its infancy could be the appropriate framework for incorporating such developments if they can be supported by subsequent experimentation on the creation of similar field structures in plasma or elsewhere.

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